BEM research
and new developments in Brazil

Ney Augusto Dumont
Civil Engineering Department – PUC-Rio

FAPERJ

CNPq
Conselho Nacional de Desenvolvimento Científico e Tecnológico

CAPES

PONTIFÍCIA UNIVERSIDADE CATÓLICA
DO RIO DE JANEIRO

BEM research and new developments in Brazil

a tiny part of

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CAPES

Pontifícia Universidade Católica do Rio de Janeiro


(A view from this tiny part of
Brazil: my office room's window)
NSF_BEM_workshop: BEM research and new developments in Brazil

Prominent BEM-researchers in Brazil

- José Cláudio Faria Telles
- Webe João Mansur
- Luiz Wrobel (at Brunel University)

And large team at COPPE/UFRJ and other universities

- Euclides de Mesquita Neto
- Paulo Sollero

At UNICAMP

- Wilson Sérgio Venturini (passed away last July)
- João Batista de Paiva
- Humberto Breves Coda

At São Carlos (UESC)

- Delfim Soares Júnior – at UFJF
- Francisco Célio de Araújo – at UFOP

New generation

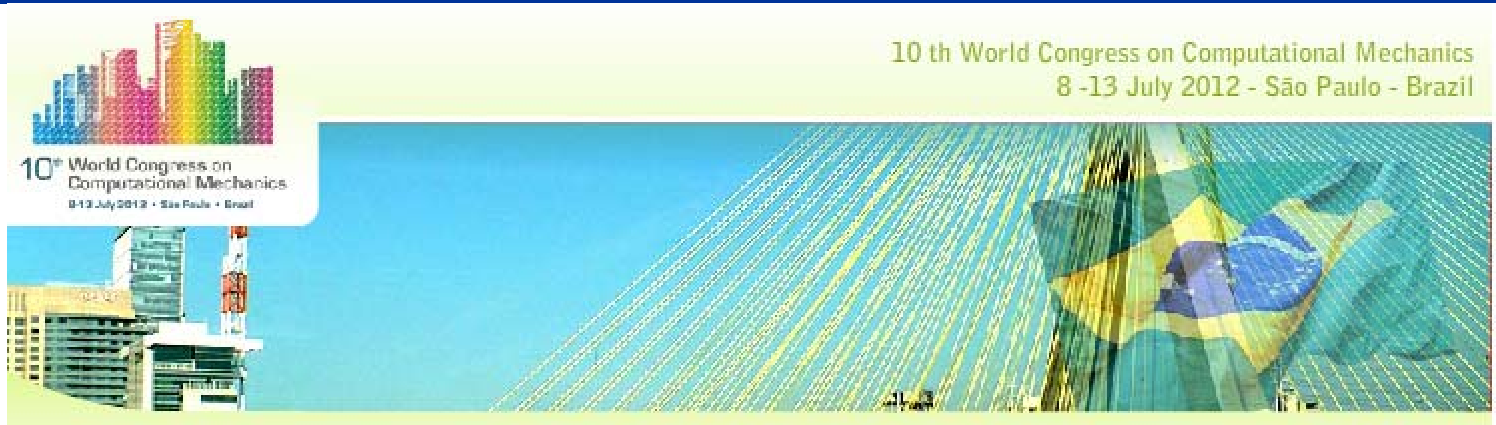
ANNOUNCEMENT

- **BETEQ 2011 - XII International Conference on Boundary Element and Meshless Techniques**

12-15 July 2011, Brasilia, Brazil

A few days in Rio de Janeiro is mandatory!

ANNOUNCEMENT



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- Hotel Accommodation
- Pre and Post Tours in Brazil
- Congress Secretariat

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www.wccm2012.com

Contact

Register Online

Propose Minisymposia

A few days in Rio de Janeiro is mandatory!

Recently graduated student and works in progress at PUC-Rio

Recently concluded Ph.D Thesis:

- M. F. F. Oliveira, *Conventional and simplied-hybrid boundary element methods applied to axisymmetric elasticity problems in fullspace and halfspace, PUC-Rio (2009). Co-advisor: Patrick Selvadurai (McGill University)*

Ph.D Theses in progress

- C. A. Aguilar M., *Comparison of the computational performance of the advanced mode superposition technique with techniques that use numerical Laplace transforms* (since September 2008).
- D. Huamán M., *Gradient elasticity formulations with the hybrid boundary element method* (since September 2008)

M.Sc. Thesis in progress

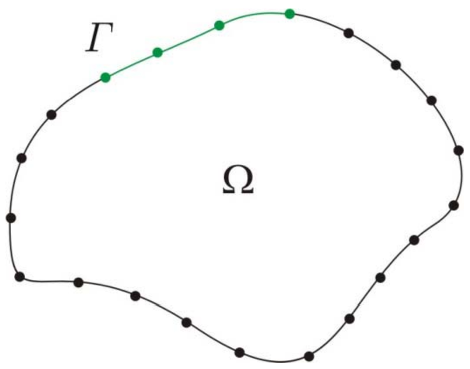
- E. Y. Mamani V., *Application of a generalized Westergaard stress function for the analysis of fracture mechanics problems* (since January 2010). Co-advisor: A. A. O. Lopes

(Intended) Contents of this presentation

- Variationally-based, hybrid boundary and finite elements
- Developments for time-dependent problems
- Developments in gradient elasticity
- Dislocation-based formulations (for fracture mechanics)
- From the collocation (conventional of hybrid) boundary element method to a meshless formulation

  Or: The expedite boundary element method

Approximations on the boundary



$$\sigma_{ji,j} + b_i = 0$$

$$\downarrow$$

$$u_i = u_{im}\, d_m$$

$$t_i = t_{i\ell}\, t_\ell$$

$$\sigma^p_{ji,j} + b_i = 0$$

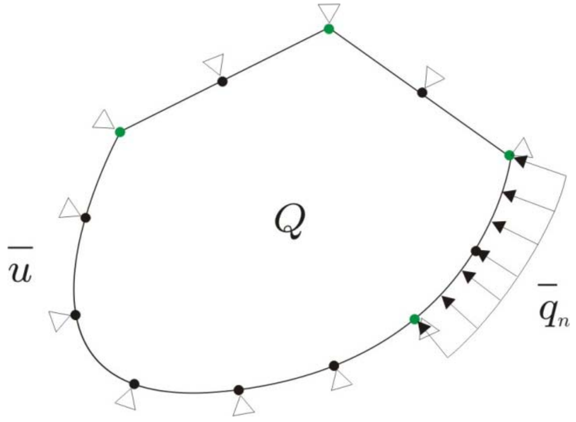
$$\downarrow$$

$$u^p_i = u_{im}\, d^p_m$$

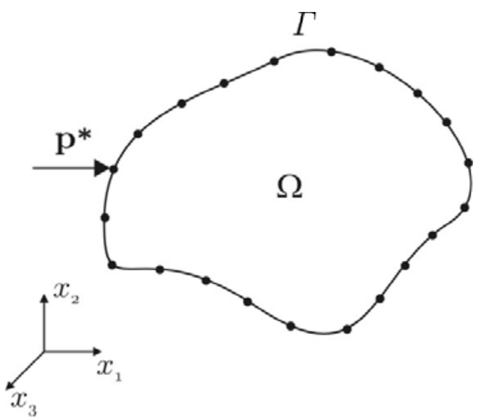
$$t^p_i = t_{i\ell}\, t^p_\ell$$

d_m, d^p_m : nodal atributes

t_ℓ, t^p_ℓ : surface atributes



Fundamental solution



$$\sigma^*_{ji,j} + \Delta_{im}\, p^*_m = 0$$

$$u^*_i = u^*_{im}\, p^*_m + \boxed{c^r_i} \equiv u^*_{im}\, p^*_m + \boxed{u^r_{is}\ C_{sm}\ p^*_m} \quad \text{in} \quad \Omega$$

$$\sigma^*_{ij} = \sigma^*_{ijm}\, p^*_m = D_{ijkp}\, u^*_{km,p}\, p^*_m \quad \text{in} \quad \Omega$$

$$t^*_i = t^*_{im}\, p^*_m = \sigma^*_{jim}\, \eta_j\, p^*_m \quad \text{on} \quad \Gamma$$

- Properties

$$\int_\Omega \sigma^*_{jim,j}\, d\Omega = -\delta_{im} \qquad \int_\Gamma t^*_{im} d\Gamma = -\delta_{im}$$

The conventional, collocation BEM

$$\mathbf{H}(\mathbf{d}-\mathbf{d}^p) \cong \mathbf{G}(\mathbf{t}-\mathbf{t}^p)$$

Derived from the Somigliana's identity

$\cong$ means congruence in terms of weighted residuals (collocation method)

(inherent approximation error due to rigid body displacements!)

$\mathbf{d}^p, \mathbf{t}^p$ are already an expedite means of taking a particular solution into account

or $$\left(\int_\Gamma \sigma^*_{jim}\eta_j u_{in}\mathrm{d}\Gamma\right)d_n \equiv H_{mn}d_n \cong G_{m\ell}t_\ell \equiv \left(\int_\Gamma u^*_{im}t_{i\ell}\mathrm{d}\Gamma\right)t_\ell$$

Where:

$d_n =$ nodal displacement attributes

$u_{in} =$ displacement interpolation functions

$t_{i\ell} =$ boundary traction force attributes

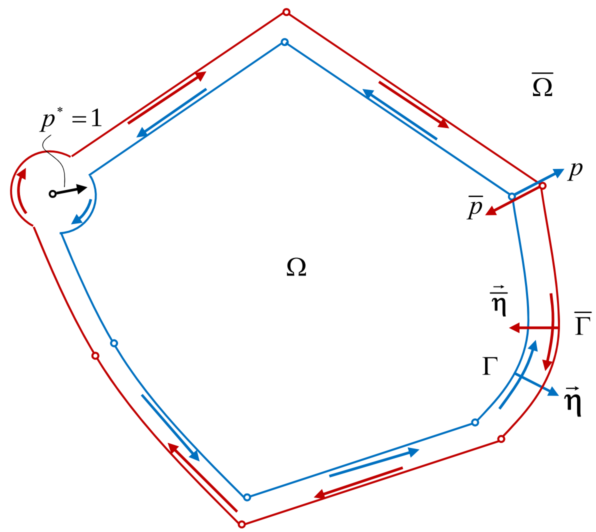
$t_{i\ell} =$ traction force interpolation functions

σ^*_{jim}, u^*_{im} Stress and displacement expressions of the problem's fundamental solution

Linear algebra properties of $\mathbf{H}$

$$\mathbf{d}^* = \mathbf{H}\mathbf{d} \quad \Leftrightarrow \quad \mathbf{p} = \mathbf{H}^{\mathrm{T}}\mathbf{p}^*; \qquad \overline{\mathbf{d}}^* = \overline{\mathbf{H}}\mathbf{d} \quad \Leftrightarrow \quad \overline{\mathbf{p}} = \overline{\mathbf{H}}^{\mathrm{T}}\mathbf{p}^*$$

$$\mathbf{d}^* + \overline{\mathbf{d}}^* = \mathbf{d}$$



$$\mathbf{H} + \overline{\mathbf{H}} = \mathbf{I}$$

$$\mathbf{p} + \overline{\mathbf{p}} = \mathbf{p}^*$$

Spectral properties:

$$\mathbf{H}\mathbf{W} = \mathbf{0}; \quad \mathbf{H}^{\mathrm{T}}\mathbf{V} = \mathbf{0}$$

$$\overline{\mathbf{H}}\mathbf{W} = \mathbf{W}; \quad \overline{\mathbf{H}}^{\mathrm{T}}\mathbf{V} = \mathbf{V}$$

Consistent formulation of the BEM

$$\mathbf{H}\left(\mathbf{d}-\mathbf{d}^{p}\right)=\mathbf{G}\mathbf{P}_{R}^{\perp}\left(\mathbf{t}-\mathbf{t}^{p}\right)$$

Where the orthogonal projector

$$\mathbf{P}_{R}^{\perp}=\mathbf{I}-\mathbf{P}_{R}=\mathbf{I}-\mathbf{R}\left(\mathbf{R}^{\mathrm{T}}\mathbf{R}\right)^{-1}\mathbf{R}^{\mathrm{T}}$$

with

$$R_{\ell s}=\int_{\Gamma} t_{i\ell} u_{is}^{r}\,\mathrm{d}\Gamma$$

comes from the fact that there is an arbitrary amount of rigid body displacements in the fundamental solution

$$u_{i}^{*}=u_{im}^{*}\;p_{m}^{*}+\boxed{c_{i}^{r}}\equiv u_{im}^{*}\;p_{m}^{*}+\boxed{u_{is}^{r}\;C_{sm}\;p_{m}^{*}}\quad \text{in}\quad \Omega$$

Literature Review – part I

Hybrid formulations: Coined by Pian in 1967 "... to signify elements which maintain either equilibrium or compatibility in the element and then to satisfy compatibility or equilibrium respectively along the interelement boundary"

Parallel developments

Hellinger, E., 1914. Die allgemeinen Ansätze der Mechanik der Kontinua, *Enz. math. Wis*, **4**, 602-694.

Reissner, E. 1950. On a variational theorem in elasticity, *J. Math. Phys.*, **29**, 90-95 .

Hu, H.-C., 1955. On some variational principles in the theory of elasticity and the theory of plasticity, *Scientia Sinica*, **4**, 33-54

Pian, T. H. H., 1964. Derivation of element stiffness matrices by assumed stress distribution. *AIAA J.,* 2, 1333-1336

Dumont, N. A., 1989. The hybrid boundary element method: an alliance between mechanical consistency and simplicity. *Applied Mechanics Reviews*, **42**, no. 11, Part 2, S54-S63

(and variations)

Trefftz, E., 1926. Ein Gegenstück zum Ritzschen Verfahren. *Proc. 2nd International Congress of Applied Mechanics*, Zurich, Switzerland.

Jirousek, J. & Leon, N., 1977. A powerful finite element for plate bending, *Com. Meths. in Appl. Mech. Engng.*, **12**, 77-96.

Qin, Q. H., 2003. *The Trefftz Finite and Boundary Element Method*, WITPress.

(Many variations)

Present theoretical investigation

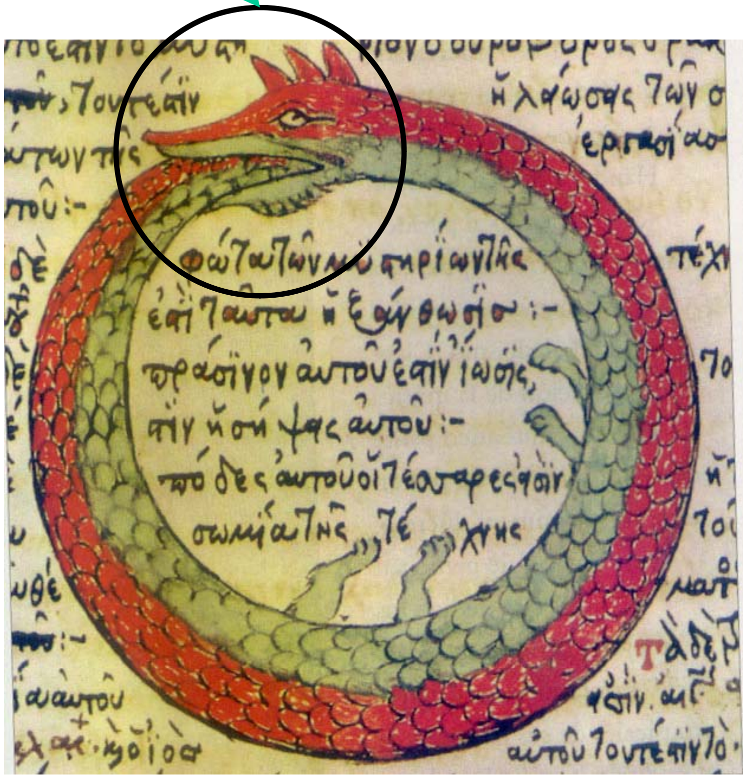
Motivation: we are here

Double layer potential matrix

$$H_{mn} = \int_{\Gamma} \sigma^{*}_{jim} \eta_j u_{in} \mathrm{d}\Gamma$$

Mechanical assumptions,
linear algebra consequences

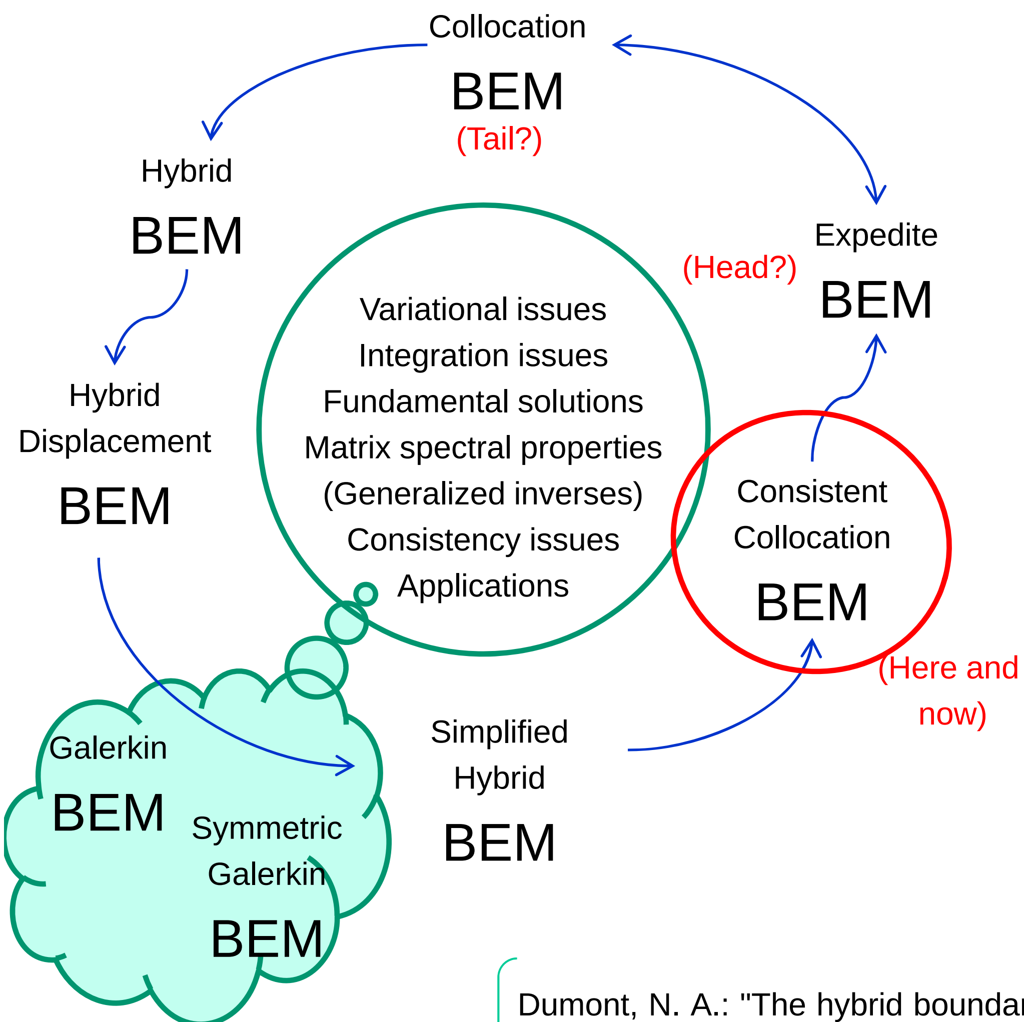
Theoretical tool:
Displacement virtual work
principle (elastostatics):



The Ouroboros (from Wikipedia)

$$\mathbf{d}^{*} = \mathbf{H}\mathbf{d} \quad \Leftrightarrow \quad \mathbf{p} = \mathbf{H}^{\mathrm{T}}\mathbf{p}^{*}$$

The Ouroboros!



(from Wikipedia)

Dumont, N. A.: "The hybrid boundary element method – fundamentals", in preparation to be submitted to Engineering Analysis with Boundary Elements

The conventional, collocation BEM

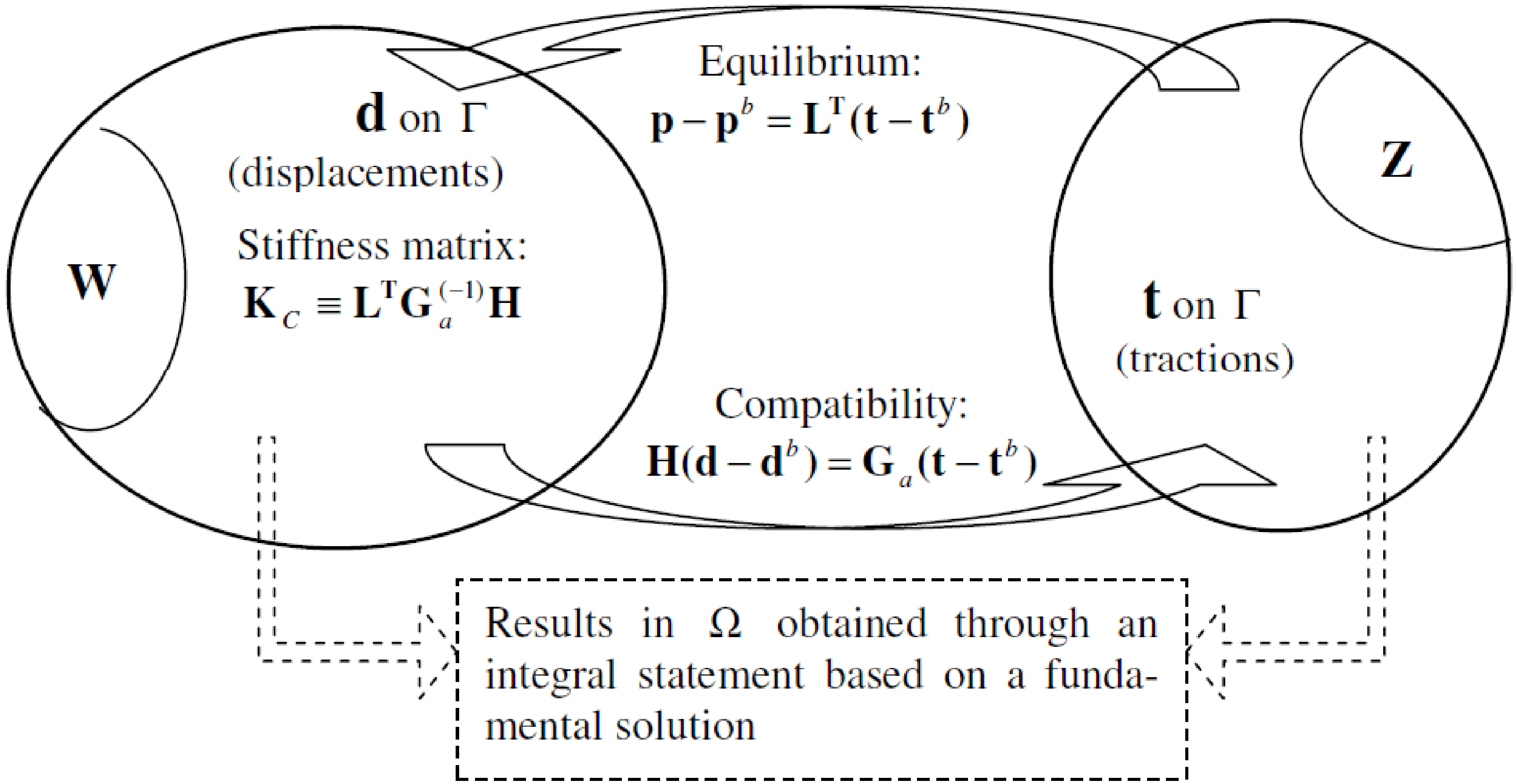


Figure 5. Transformations carried out in the conventional boundary element method.

The hybrid BEM

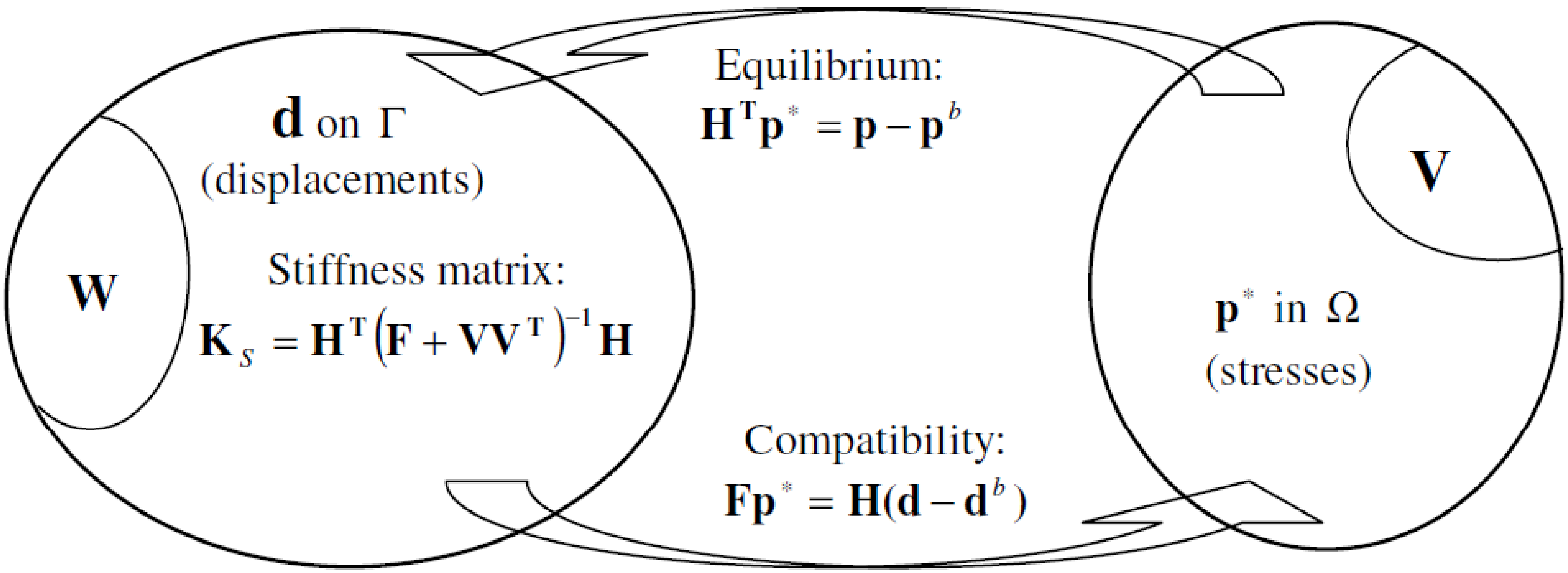


Figure 7. Transformations carried out in the hybrid stress boundary element method.

Dumont, N. A.: “Variationally-Based, Hybrid Boundary Element Methods”,
Computer Assisted Mechanics and Engineering Sciences (CAMES) Vol 10 pp 407-430, **2003**

The hybrid displacement BEM

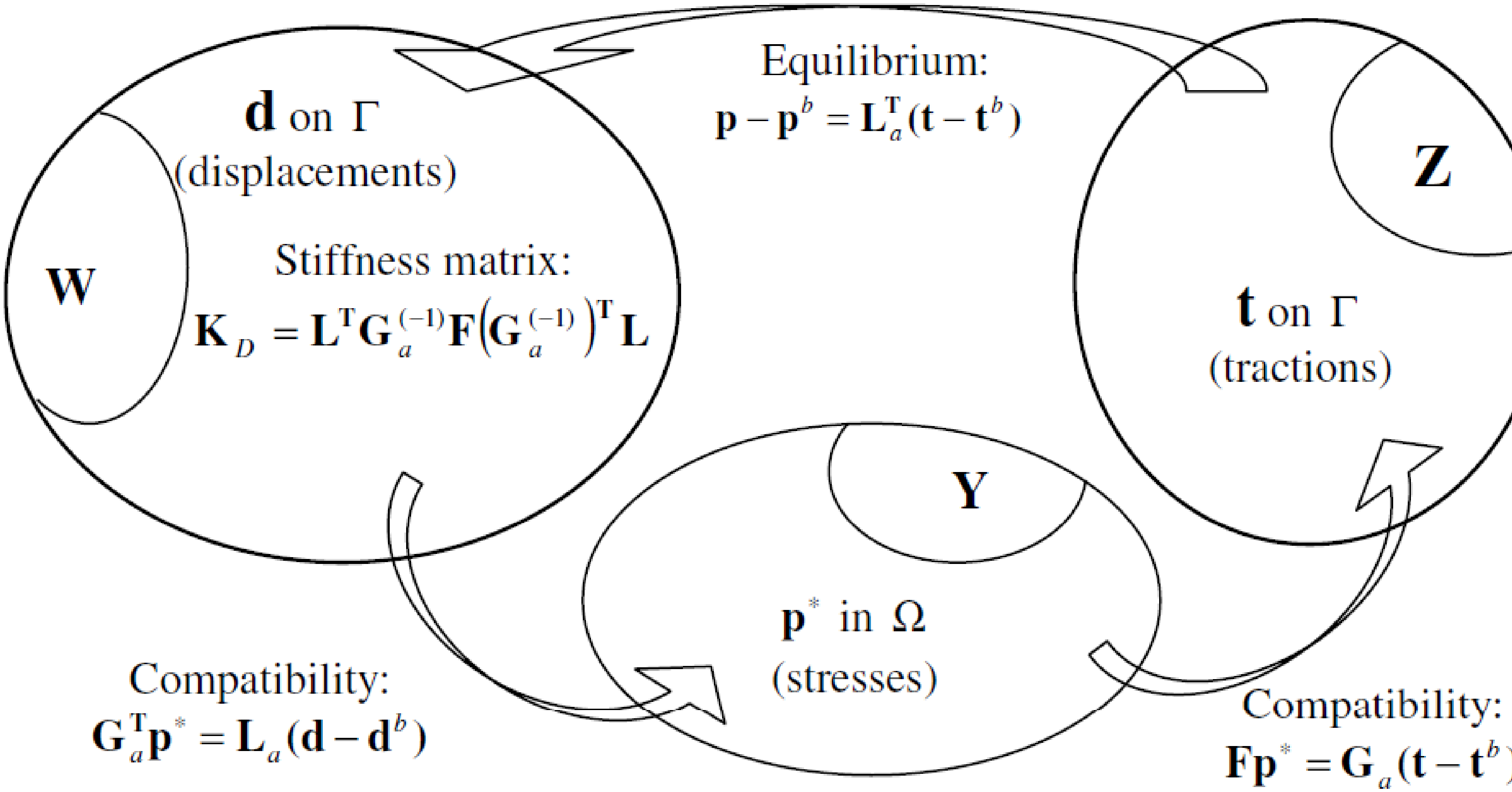


Figure 6. Transformations carried out in the hybrid displacement boundary element method.

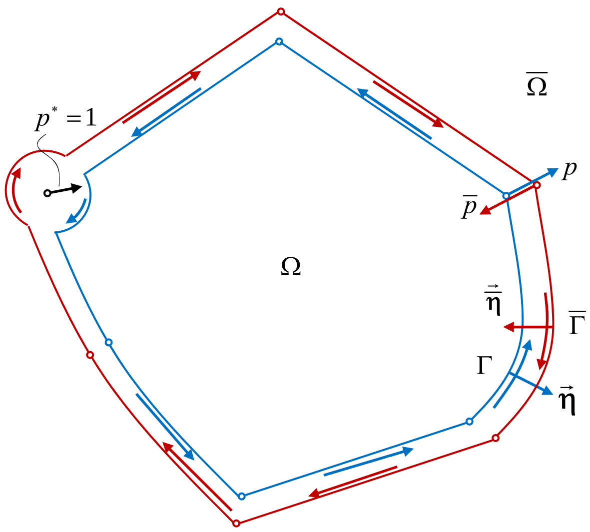
Definition of the matrices involved

$$H_{mn} := \int_{\Gamma} \sigma^{*}_{jim} \eta_j u_{in} \mathrm{d}\Gamma$$

$$G_{m\ell} := \int_{\Gamma} u^{*}_{im} t_{i\ell} \mathrm{d}\Gamma$$

$$F^{*}_{mn} := \int_{\Gamma} \sigma^{*}_{jim} \eta_j u^{*}_{in} \mathrm{d}\Gamma$$

$$L_{\ell m} := \int_{\Gamma} u_{im} t_{i\ell} \mathrm{d}\Gamma$$



Simplified HBEM: $\mathbf{F}^{*} \leftarrow \mathbf{H}\mathbf{U}^{*}$

The Ouroboros!

Some isolated virtual work statements

$$\mathbf{H}^{\mathrm{T}}\mathbf{p}^{*} = \mathbf{p} \quad \Leftrightarrow \quad \mathbf{H}\mathbf{d} = \mathbf{d}^{*}$$

$$\mathbf{G}\mathbf{t} = \mathbf{d}^{*} \quad \Leftrightarrow \quad \mathbf{G}^{\mathrm{T}}\mathbf{p}^{*} = \mathbf{d}^{L}$$

$$\mathbf{L}^{\mathrm{T}}\mathbf{t} = \mathbf{p} \quad \Leftrightarrow \quad \mathbf{L}\mathbf{d} = \mathbf{d}^{L}$$

There are more virtual work statements;
Several restrictions apply!

$(\mathbf{d}, \mathbf{p})$ External reference system

$(\mathbf{p}^{*}, \mathbf{d}^{*})$ Internal reference system

$(\mathbf{t}, \mathbf{d}^{L})$ System of Lagrange multipliers

G and **L** should be rectangular, in general;
G should be consistent

Dumont, N.A., An assessment of the spectral properties of the matrix G used in the boundary element methods, *Computational Mechanics*, **22**(1), pp. 32-41, 1998

The Ouroboros!

Some isolated virtual work statements

$$\mathbf{H}^{\mathrm{T}}\mathbf{p}^{*} = \mathbf{p} \quad \Leftrightarrow \quad \mathbf{H}\mathbf{d} = \mathbf{d}^{*}$$

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$\left(\mathbf{p}^{*}, \mathbf{d}^{*}\right)$ Internal reference system

$\left(\mathbf{t}, \mathbf{d}^{L}\right)$ System of Lagrange multipliers

$\mathbf{G}$ and $\mathbf{L}$ should be rectangular, in general;
$\mathbf{G}$ should be consistent

$$\mathbf{F}^{*}\mathbf{p}^{*} = \mathbf{d}^{*} \qquad \mathbf{K}\mathbf{d} = \mathbf{p}$$

Results at internal points: no integral statement actually required

Spectral properties for the hybrid BEM

For a finite domain:

$\mathbf{P}_W$ is the orthogonal projector onto the space of rigid body displacements and $\mathbf{P}_W^\perp = \mathbf{I} - \mathbf{P}_W$

Then, $\mathbf{H}\mathbf{P}_W = \mathbf{0} \quad \Rightarrow \quad \mathbf{H}^\mathrm{T}\mathbf{P}_V = \mathbf{0}$ ⟵ Definition of $\mathbf{P}_V$

For consistency, $\mathbf{F}^*\mathbf{P}_V = \mathbf{0}$

and $\mathbf{K} = \mathbf{H}^\mathrm{T}\left(\mathbf{F}^* + \mathbf{P}_V\right)^{-1}\mathbf{H}$ in $\mathbf{K}\mathbf{d} = \mathbf{p}$

Results at internal points: no integral statement actually required!

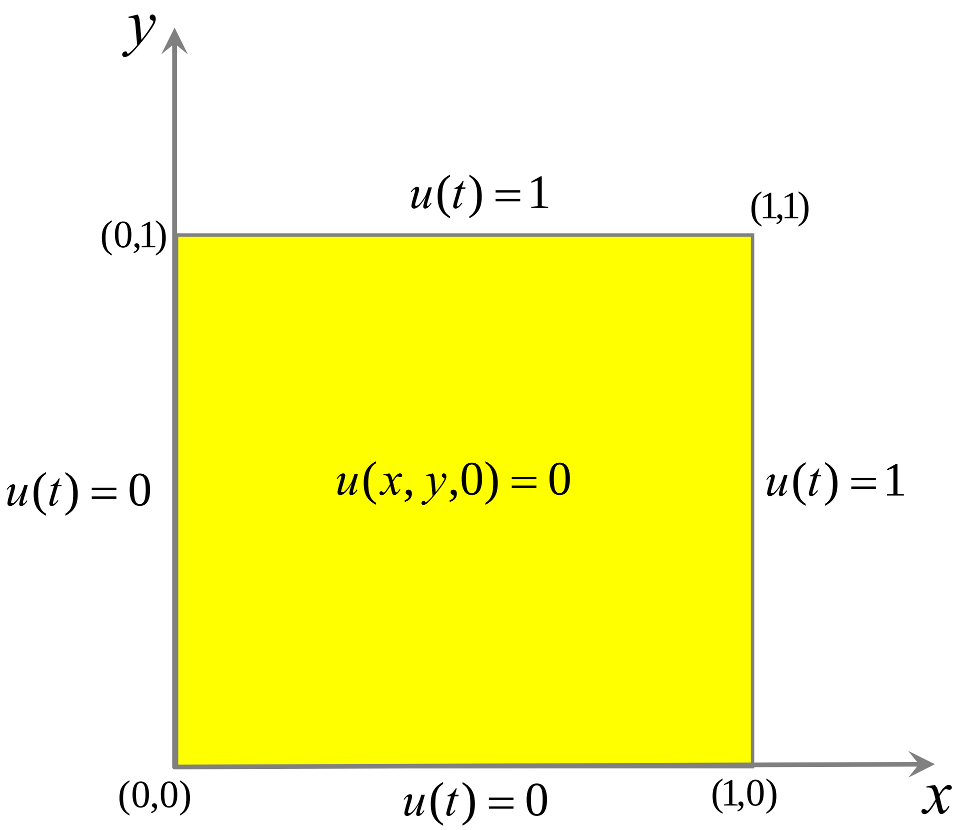
$\mathbf{p}^*$ solved from either $\mathbf{F}^*\mathbf{p}^* = \mathbf{H}\mathbf{d}$ or $\mathbf{H}^\mathrm{T}\mathbf{p}^* = \mathbf{p}$ with $\mathbf{P}_V\mathbf{p}^* = \mathbf{0}$

and $u_i^* = u_{im}^* \, p_m^* + c_i^r$ $\qquad \sigma_{ij}^* = \sigma_{ijm}^* \, p_m^*$

Some applications already implemented

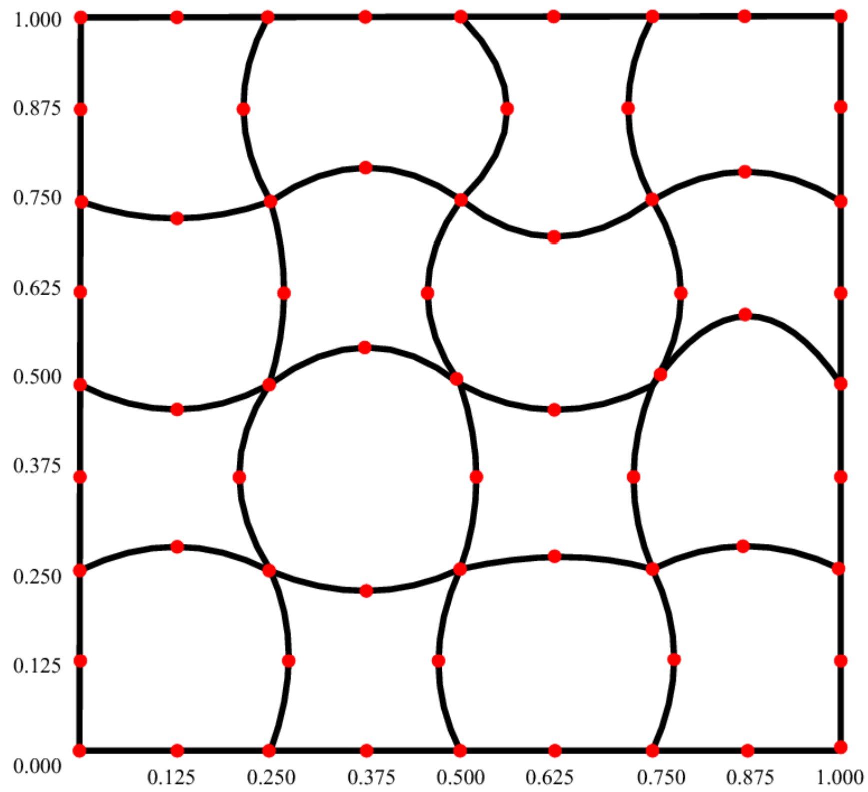
- 2D and 3D potential and elasticity problems
- 2D and 3D acoustics
- 2D general time-dependent problems (in the frequency domain):
  - Advanced mode superposition analysis
  - Use of numerical inverse transforms
- 2D sensitivity analysis
- 2D FGM-analysis for potential problems
- 2D fracture-mechanics analysis – evaluation of stress intensity factors
- Axysimmetric fullspace and halfspace elastostatics
- 2D (and 3D) gradient elasticity analysis

Example of application of non-singular fundamental solutions: two dimensional transient heat conduction in a homogeneous square plate



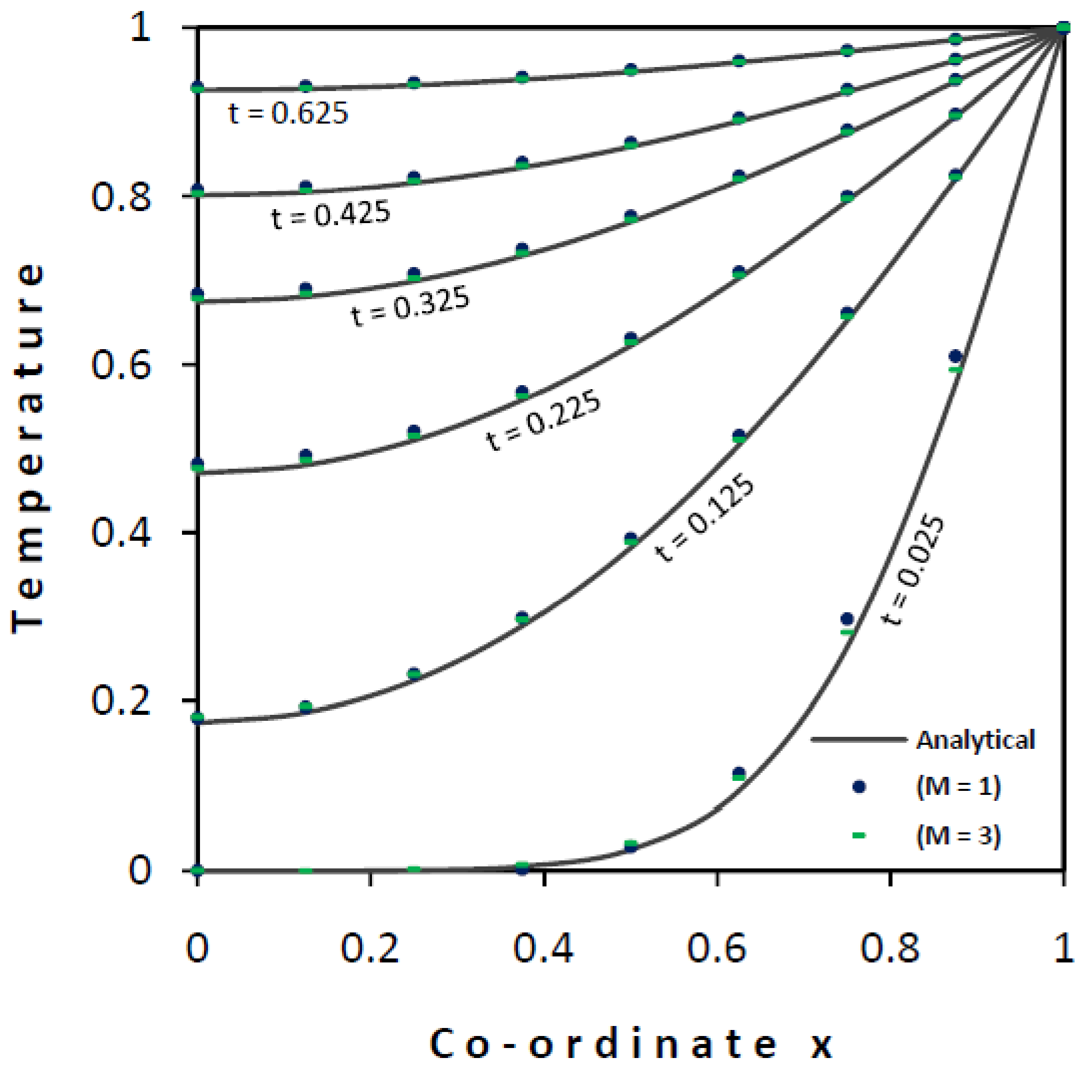
$k=1$ Conductivity

$c=1$ Specific heat



Distorced 4x4 mesh

Temperature results along the edge x = 0 for several time instants



Analytic solution

Classical modal analysis.

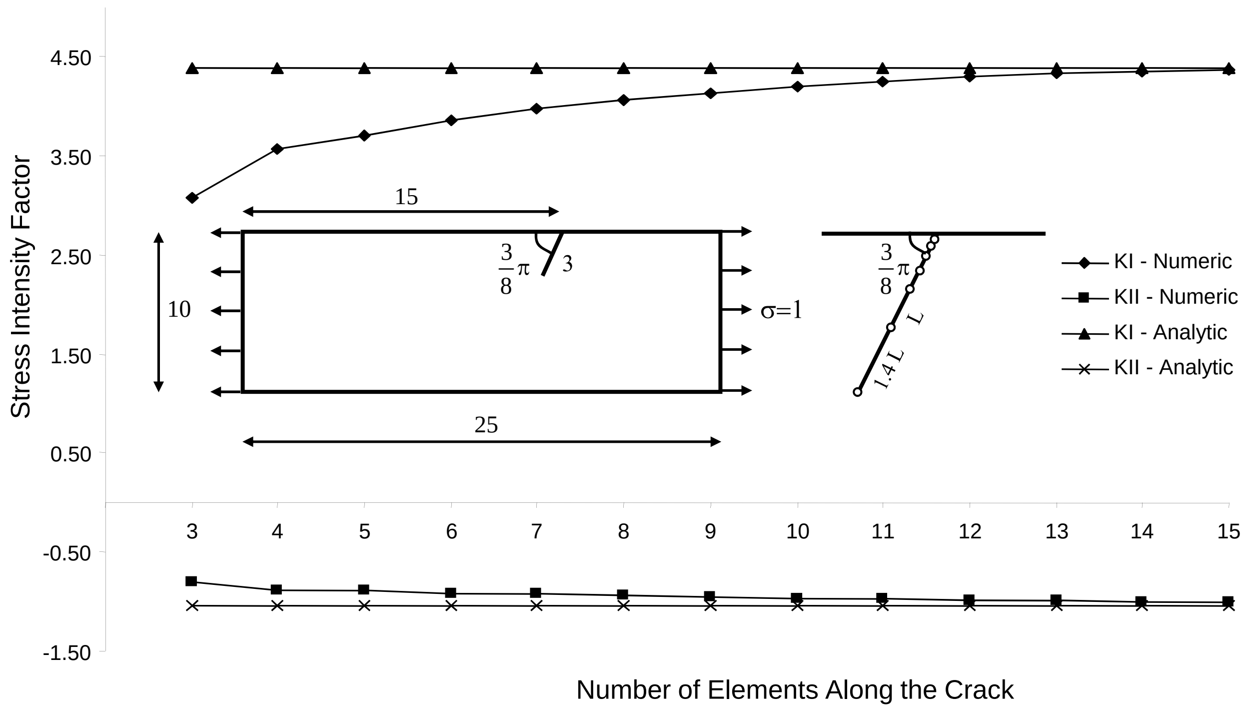
Serendipity elements.
Quadratic elements.

Hybrid formulation

3 mass matrices.

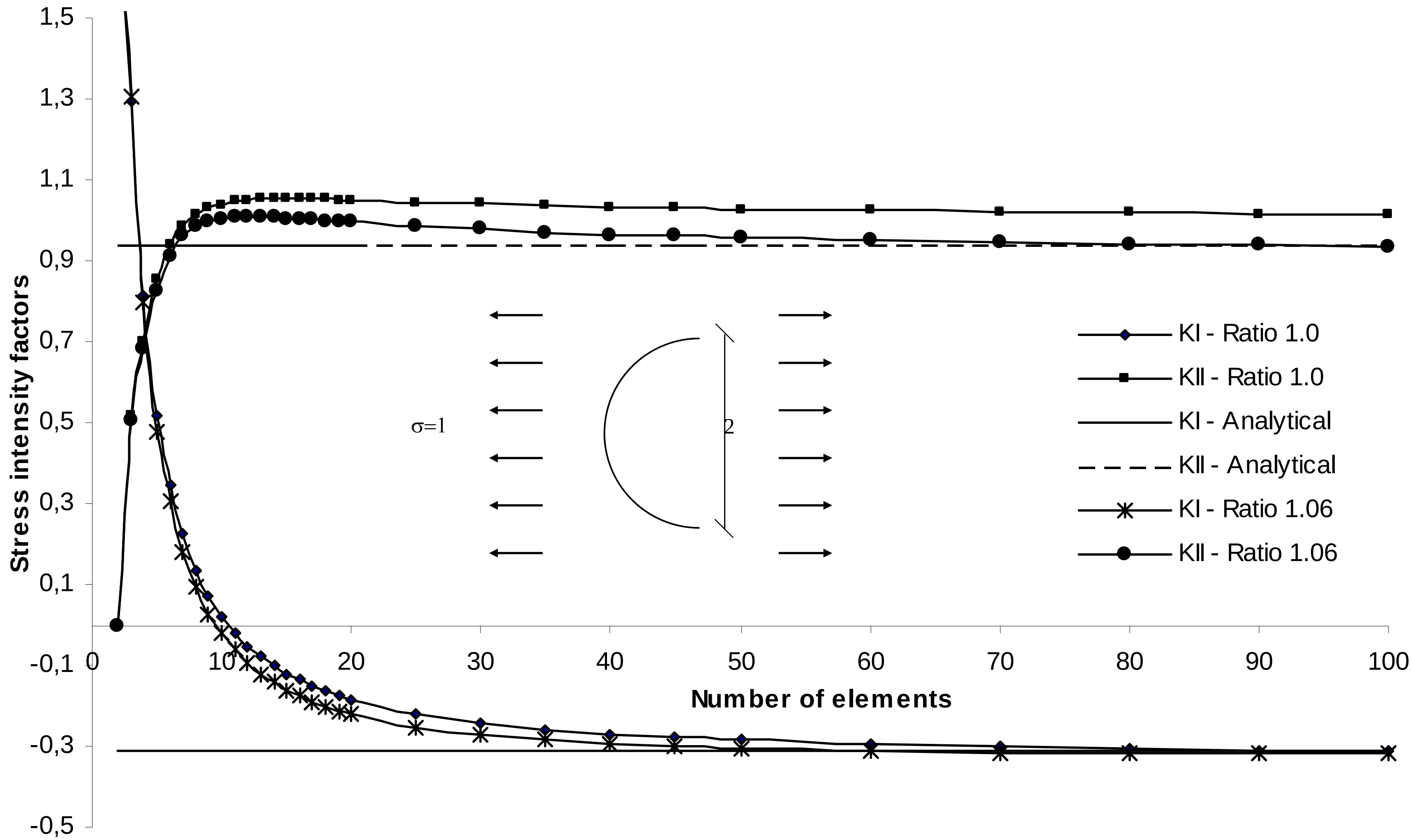
Westergaard's Stress Function as a Fundamental Solution

- Example 8 – inclined edge crack in a rectangular plate



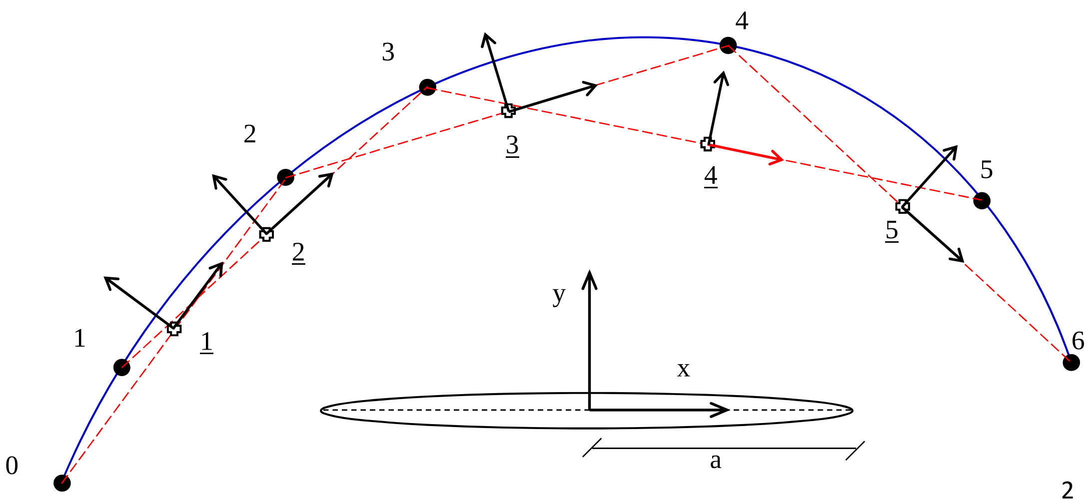
Westergaard's Stress Function as a Fundamental Solution

Example 3 – semicircular crack in an infinite continuum

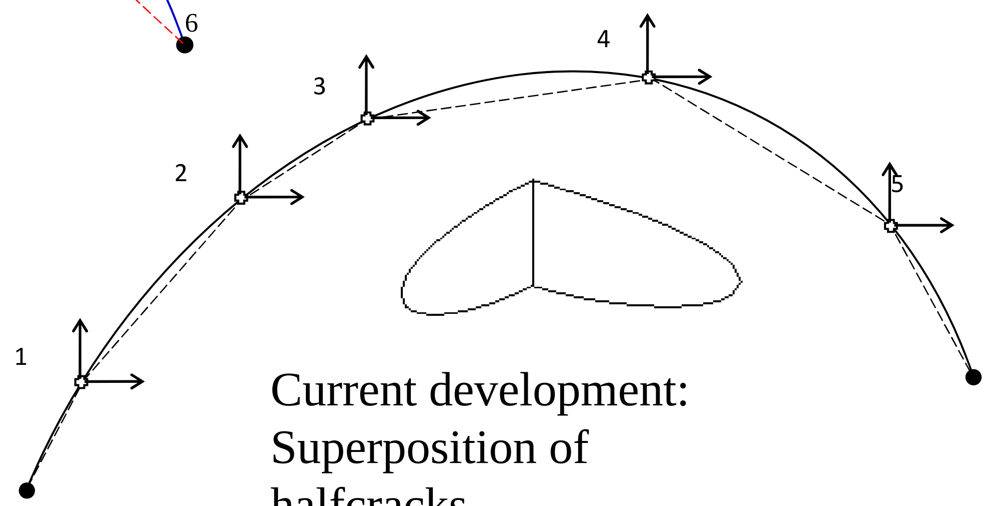


Westergaard's Stress Function as a Fundamental Solution

Modeling a curved crack



Previous development:
superposition of elipsis

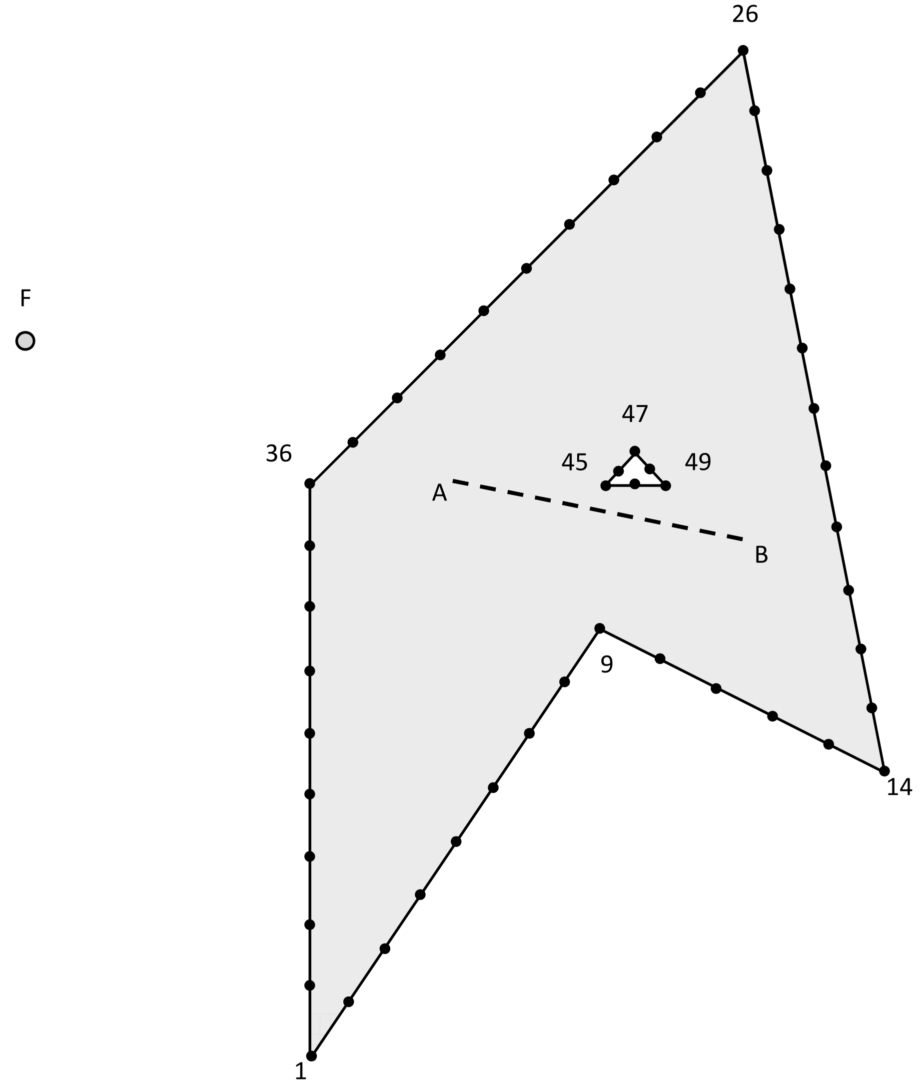


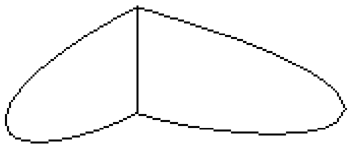
Current development:
Superposition of
halfcracks

Inspired by:

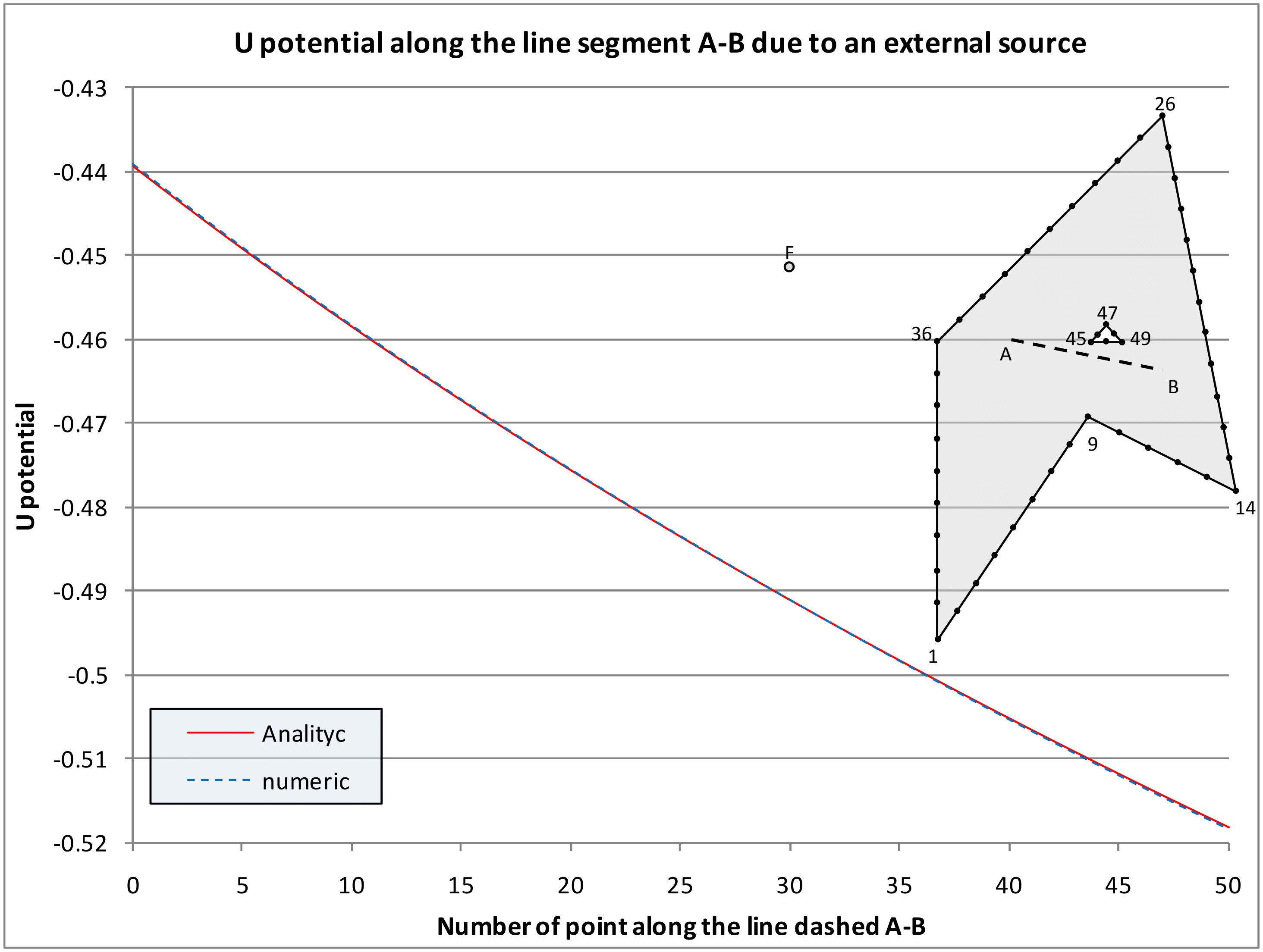
Tada, H., Ernst, H. A., Paris, P. C. (1993), Westergaard stress functions for displacement-prescribed crack problems – I, Int. Journal of Fracture, Vol. 61, pp 39-53

Westergaard's Stress Function as a Fundamental Solution

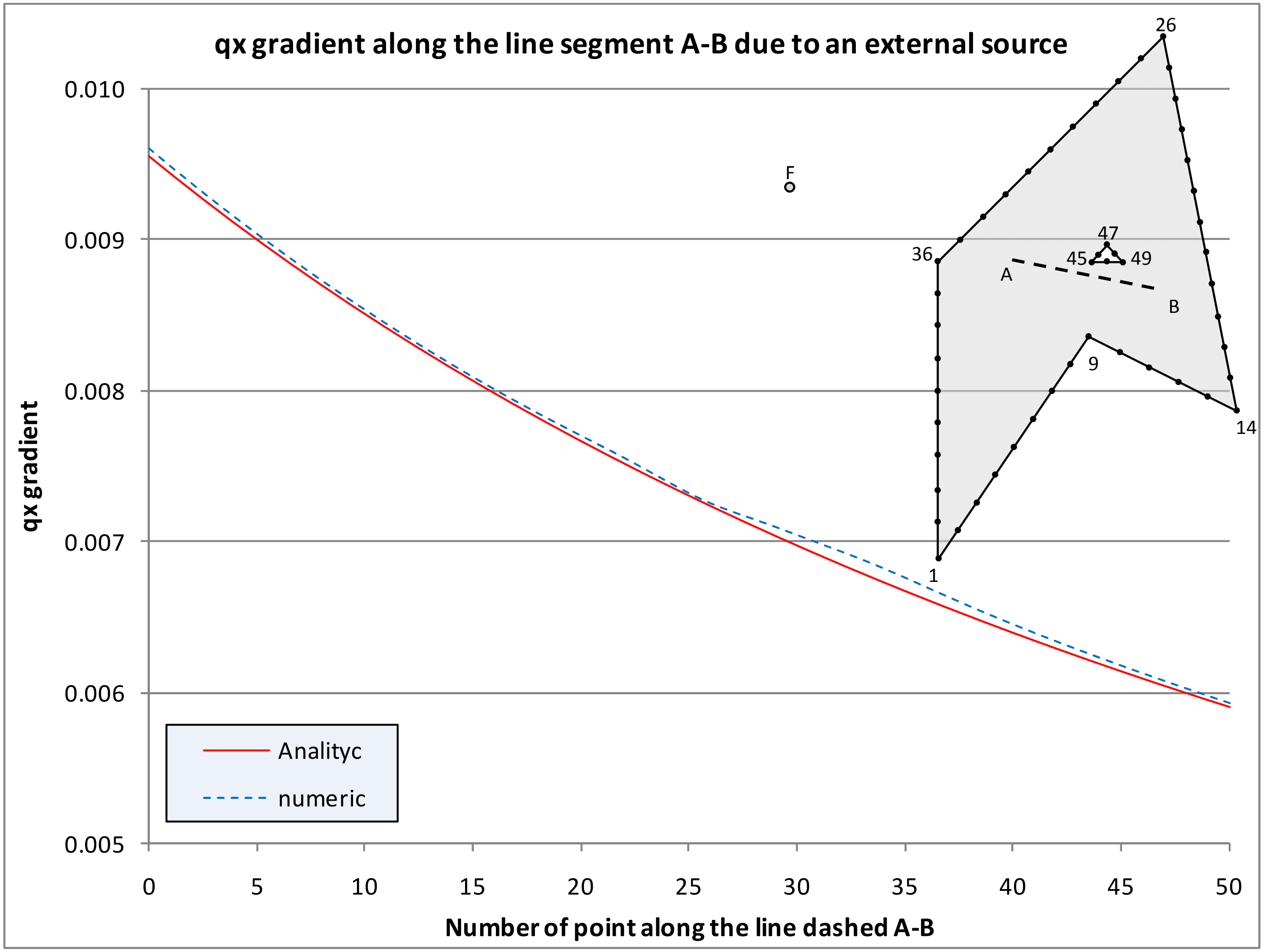




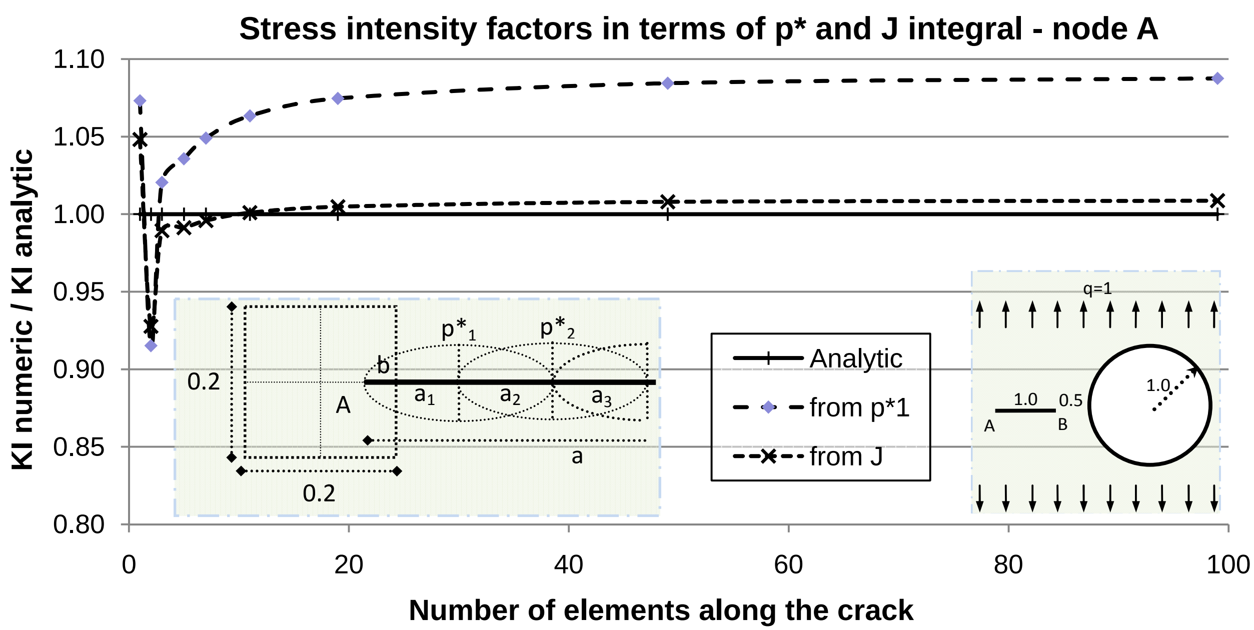
Westergaard's Stress Function as a Fundamental Solution



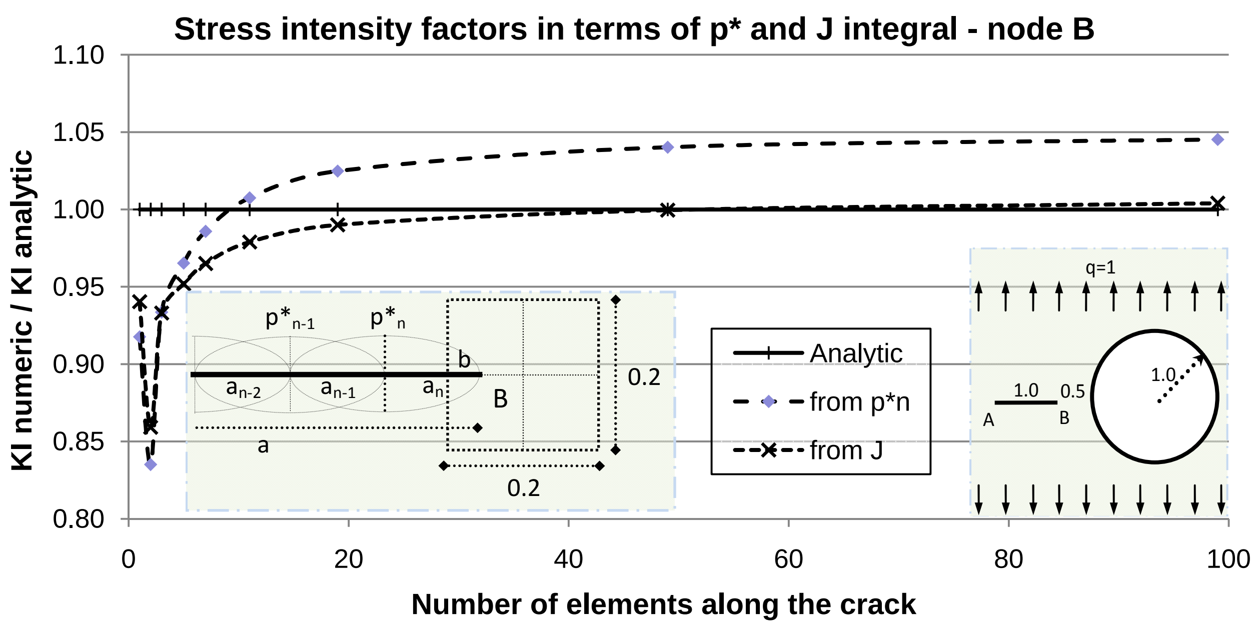
Westergaard's Stress Function as a Fundamental Solution



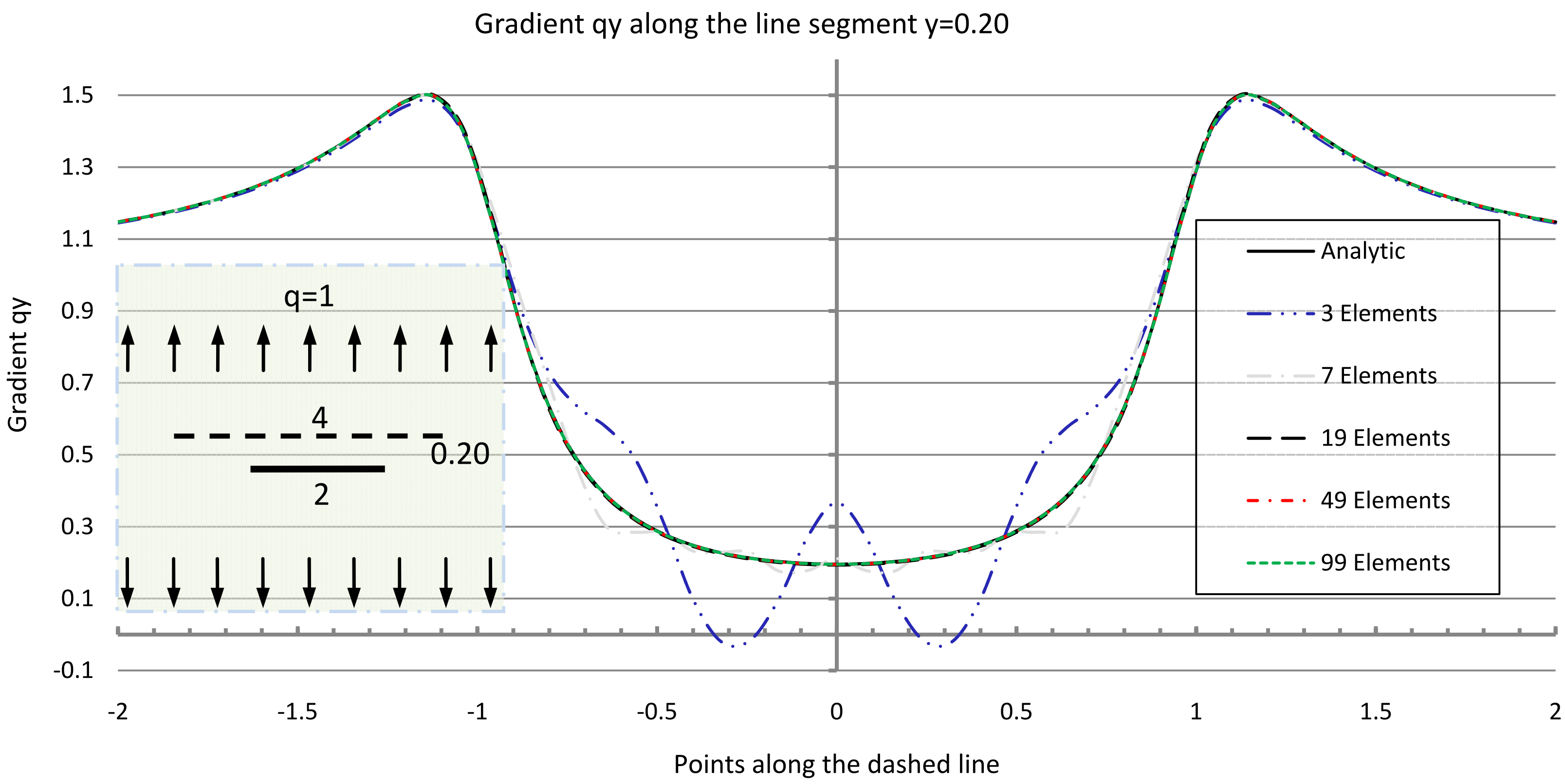
Westergaard's Stress Function as a Fundamental Solution



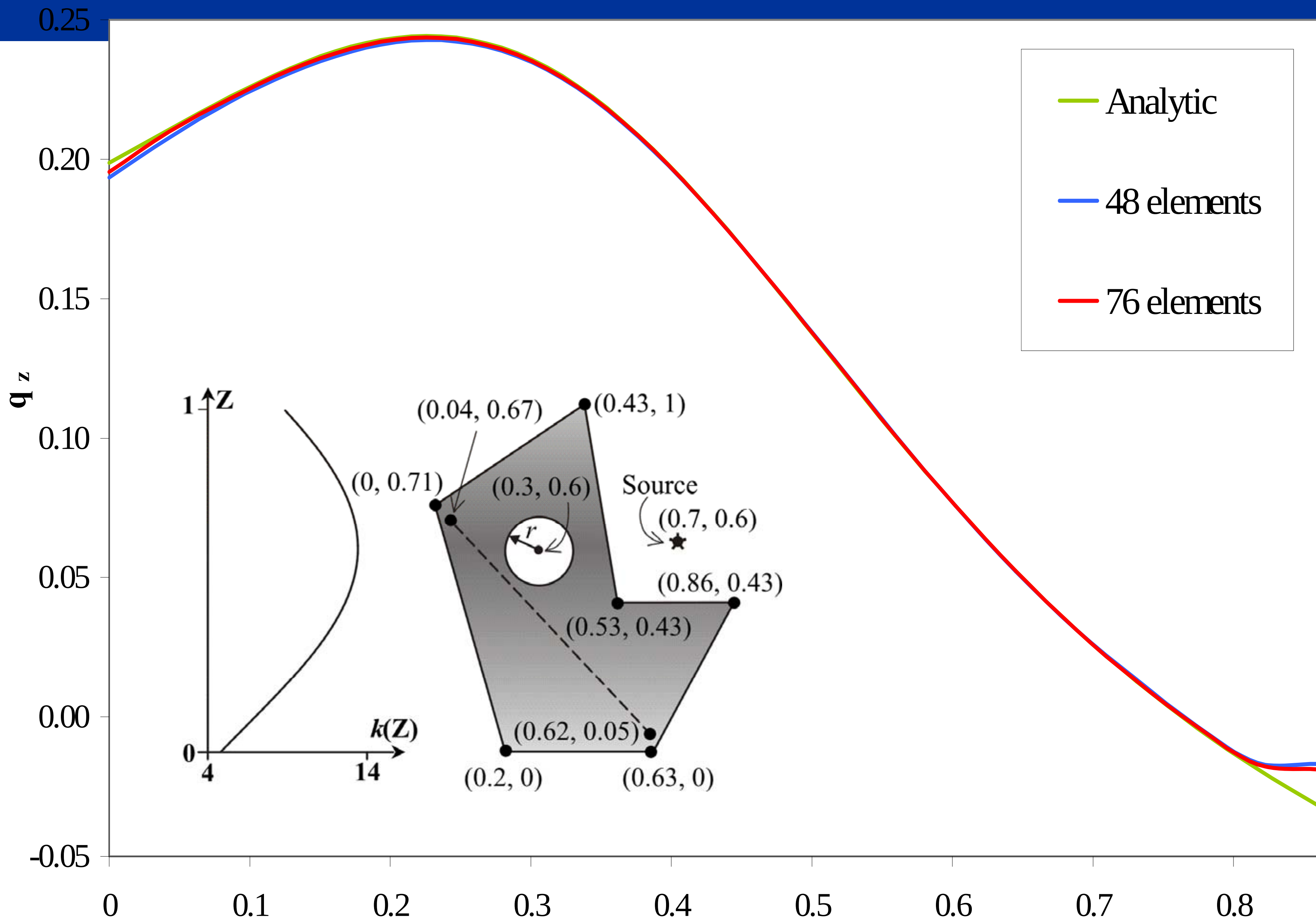
Westergaard's Stress Function as a Fundamental Solution



Westergaard's Stress Function as a Fundamental Solution



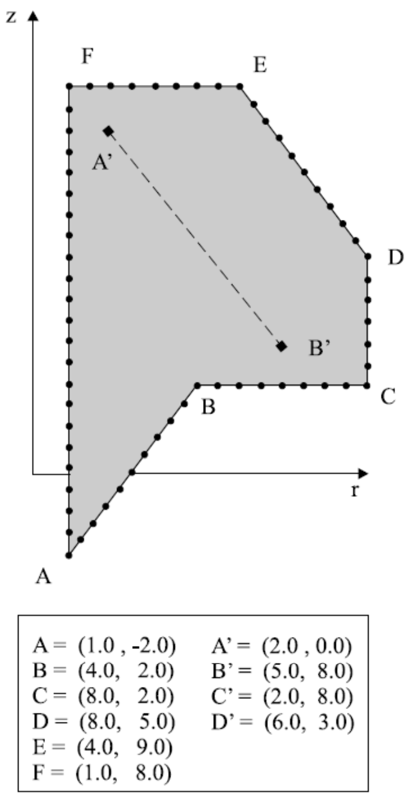
EXAMPLE: gradient q_z along a line segment

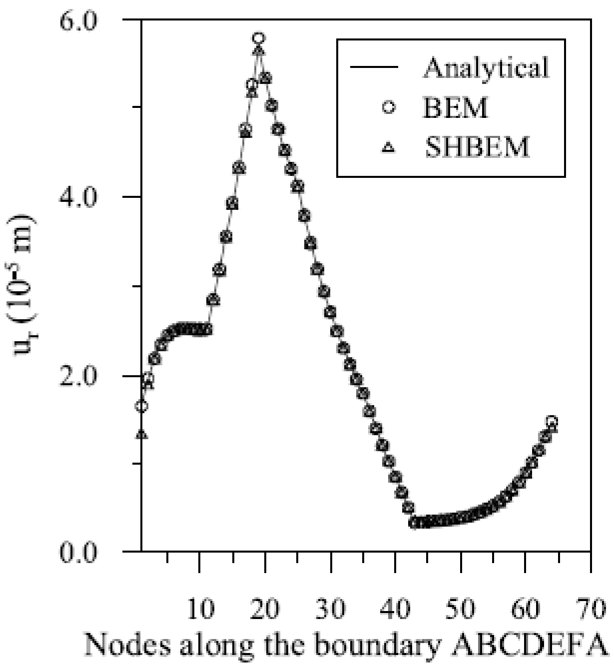


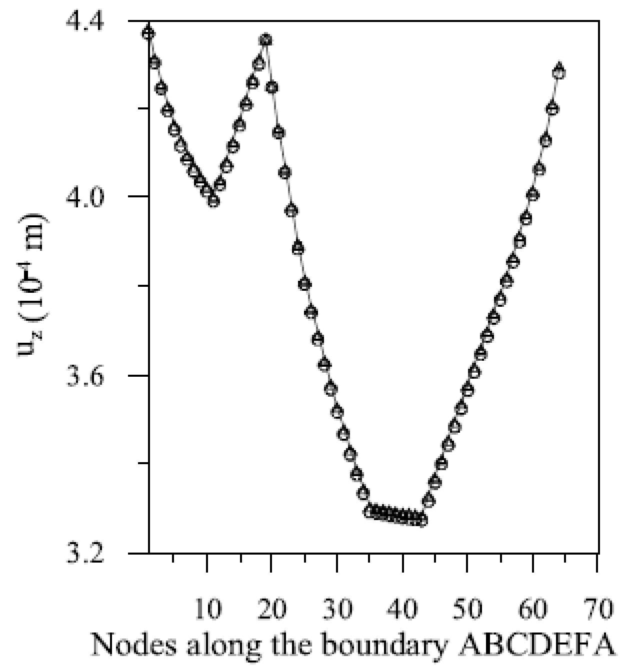
R. A. P. Chaves, "The Hybrid Simplified Boundary Element Method Applied to Time-Dependent Problems", Ph.D. Thesis, PUC-Rio, Brazil, 2003

Numerical example

Non-convex axisymmetric volume with multiply connected surfaces





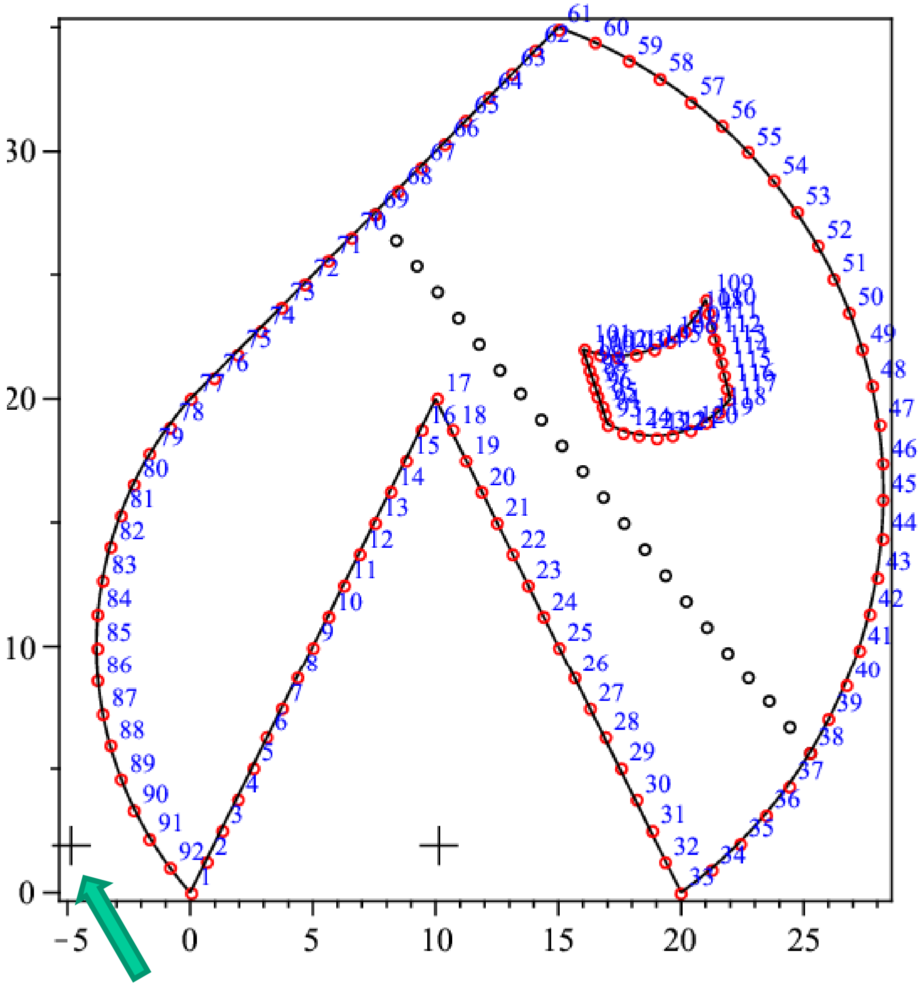


- Unitary ring sources (10, -5)

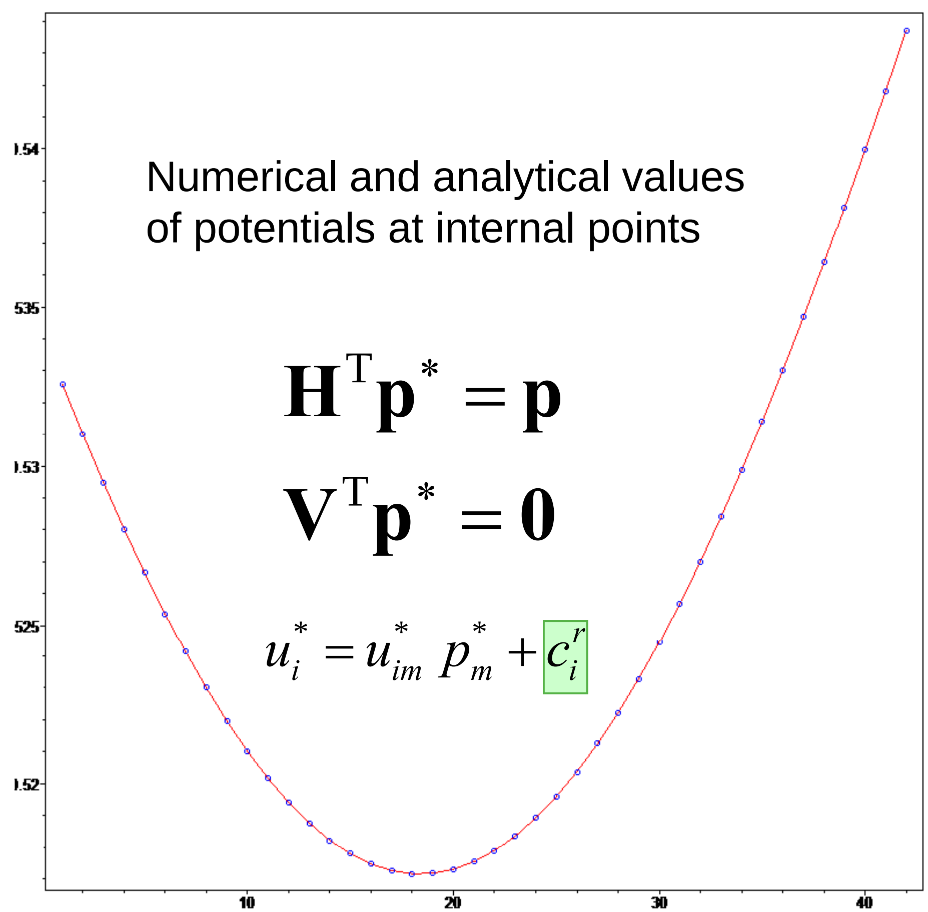
An expedite formulation of the BEM

Some numerical assessments for 2D potential problems

Irregularly shaped structure with 124 nodes



sol_9: $u(x,y) = \dfrac{\ln\left((x-x_1)^2 + (y-y_1)^2\right)}{4\pi}$



Numerical and analytical values of potentials at internal points

$$\mathbf{H}^{\mathrm{T}}\mathbf{p}^* = \mathbf{p}$$

$$\mathbf{V}^{\mathrm{T}}\mathbf{p}^* = \mathbf{0}$$

$$u_i^* = u_{im}^* \, p_m^* + \boxed{c_i^r}$$

An expedite formulation of the BEM

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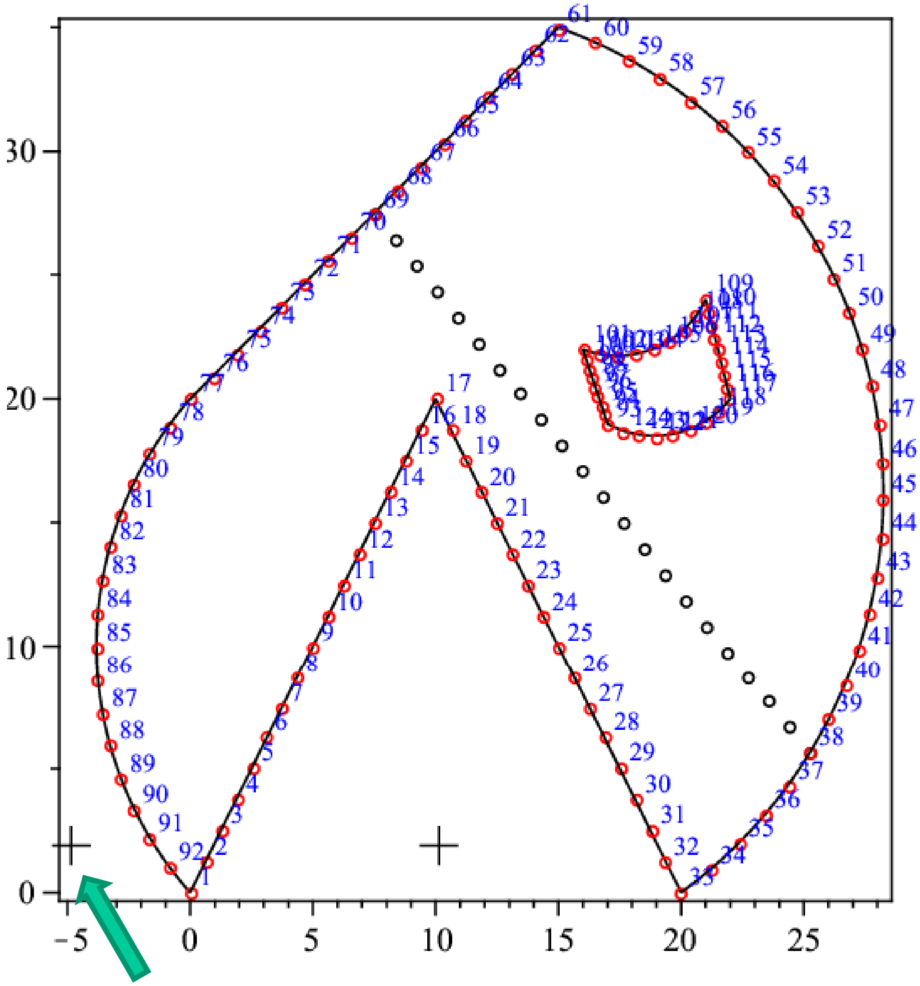

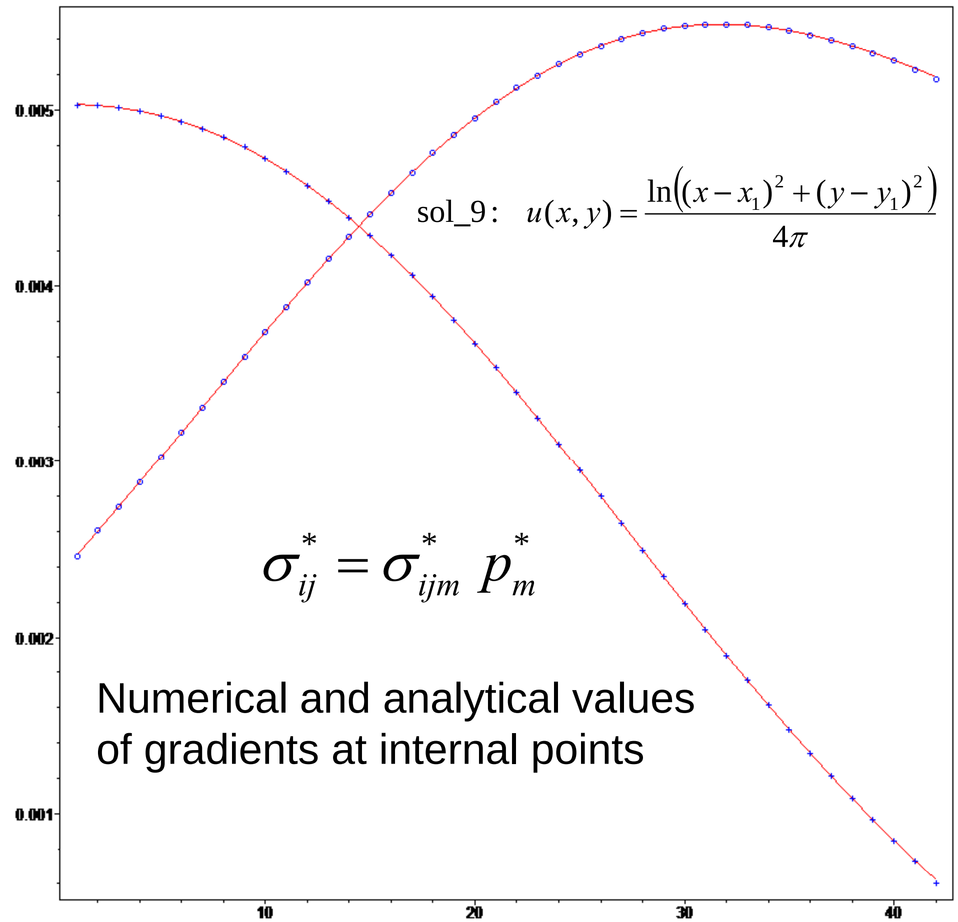

Contents of this presentation

- Variationally-based, hybrid boundary and finite elements
- Developments for time-dependent problems
- Developments in gradient elasticity
- Dislocation-based formulations (for fracture mechanics)
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  Or: The expedite boundary element method

GENERALIZED HELLINGER-REISSNER POTENTIAL

$$\int_{t_0}^{t_1}\left(-\int_\Omega\left(\delta\sigma_{ij},_j-\rho\delta\ddot{u}_i\right)\left(u_i-\tilde{u}_i\right)d\Omega+\int_\Gamma\delta\sigma_{ij}\eta_j\left(u_i-\tilde{u}_i\right)d\Gamma+\right.$$

$$\left.+\int_\Omega\delta\tilde{u}_i\left(\sigma_{ij},_j+\bar{f}_i-\rho\ddot{u}_i\right)d\Omega-\int_\Gamma\delta\tilde{u}_i\left(\sigma_{ij}\eta_j-\bar{t}_i\right)d\Gamma\right)dt=0$$

$\delta\tilde{u}_i=0$ along Γ_u

$\delta u_i=0$ at both time interval extremities t_0 and t_1

Literature Review – part I (continued)

Przemieniecki, J.S. (1968). *Theory of Matrix Structural Analysis*, Dover Publications, New York
(displacement-based free vibration analysis – truss and beam):

$$\mathbf{K} = \mathbf{K}_0 - \omega^2 \mathbf{M}_0 - \omega^4 \left(\mathbf{M}_2 - \mathbf{K}_4 \right) - \omega^6 \left(\mathbf{M}_4 - \mathbf{K}_6 \right) + O(\omega^8)$$

Voss, H., 1987, A New Justification of Finite Dynamic Element Methods, *Numerical Treatment of Eigenvalue Problems*, Vol. 4, 232-242, eds. J. Albrecht, L. Collatz, W. Velte, W. Wunderlich, Int. Series on Num. Maths. 83, Birkhäuser Verlag, Stuttgart

Gupta, Paz: several other displacement-based elements for plane-state problems. Coined *finite dynamic elements*, although they only have been applied to free vibration analysis (better, then: *finite harmonic elements*!)

?

$$\mathbf{K} = \mathbf{K}_0 - \omega^2 \left(\mathbf{M}_0 - \mathbf{K}_2 \right) - \omega^4 \left(\mathbf{M}_2 - \mathbf{K}_4 \right) - \omega^6 \left(\mathbf{M}_4 - \mathbf{K}_6 \right) + O(\omega^8)$$

Directly based on Pian and Przemieniecki: **General , consistent finite and boundary dynamic element families: acoustics, free vibration, transient problems via an advanced modal analysis.**

Dumont, N. A. & Oliveira, R., (1993) 1997. The exact dynamic formulation of the hybrid boundary element method, *Procs. XVIII CILAMCE*, Brasília, 29 - 31 October, 357-364

Dumont, N. A. & Oliveira, R., 2001. From frequency-dependent mass and stiffness matrices to the dynamic response of elastic systems. Int. J. Sol. Struct., **38**, 10-13, 1813-1830

Literature Review – part II

Lancaster, P., 2002. *Lambda-Matrices & Vibrating Systems*, Dover Publications (from 1966)

$$(\mathbf{F}_0 + \omega^2 \mathbf{F}_1 + \omega^4 \mathbf{F}_2 + \ldots)\mathbf{p}^* = (\mathbf{H}_0 + \omega^2 \mathbf{H}_1 + \omega^4 \mathbf{H}_2 + \ldots)\mathbf{d}$$

$$(\mathbf{H}_0^T + \omega^2 \mathbf{H}_1^T + \omega^4 \mathbf{H}_2^T + \ldots)\mathbf{p}^* = \mathbf{p}$$

Dumont, N. A.: “On the Inverse of Generalized Lambda Matrices with Singular Leading Term”, IJNME, Vol 66(4), 571-603, **2006**

$$(\mathbf{K}_0 - \omega^2 \mathbf{M}_1 - \omega^4 \mathbf{M}_2 + \ldots)\mathbf{d} = \mathbf{p}(\omega)$$

(Przemieniecki)

$$\left\{ \mathbf{H}_0^T \mathbf{V} = \mathbf{0} \qquad \mathbf{F}_0 \mathbf{V} = \mathbf{0} \qquad \mathbf{H}^T \mathbf{F}^{-1} \mathbf{H} = \mathbf{K} \right\}$$

Nonlinear eigenvalue problem:

Dumont, N. A.: “On the solution of generalized non-linear complex-symmetric eigenvalue problems”, IJNME, Vol 71, 1534-1568, **2007**

Sleipjen, G. L. G., Van der Horst, H. A., 1996, A Jacobi-Davidson iteration method for linear eigenvalue problems, *SIAM J. Matrix Anal. Appl*., 17, 401-425

Arbenz, P, Hochstenbach, M. E., 2004. A Jacobi-Davidson method for solving complex-symmetric eigenvalue problems, *SIAM J. Sci. Comput*. 25, 5, 1655-1673

Problems with viscous damping

Nonlinear eigenproblem:

$$\mathbf{K}_0\boldsymbol{\Phi} - \sum_{j=1}^{n}\left(i\mathbf{C}_j\boldsymbol{\Phi}\boldsymbol{\Omega}^{2j-1} + \mathbf{M}_j\boldsymbol{\Phi}\boldsymbol{\Omega}^{2j}\right) = \mathbf{0}$$

Augmented formulation:

$$\begin{bmatrix} \mathbf{K}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_1 & i\mathbf{C}_2 & \mathbf{M}_2 & \cdots & \mathbf{M}_n \\ \mathbf{0} & i\mathbf{C}_2 & \mathbf{M}_2 & i\mathbf{C}_3 & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & i\mathbf{C}_3 & \ddots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{M}_n & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0,n-1} \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n-1,0} & \Phi_{n-1,1} & \cdots & \Phi_{n-1,n-1} \end{bmatrix}$$

$$- \begin{bmatrix} i\mathbf{C}_1 & \mathbf{M}_1 & i\mathbf{C}_2 & \mathbf{M}_2 & \cdots & \mathbf{M}_n \\ \mathbf{M}_1 & i\mathbf{C}_2 & \mathbf{M}_2 & i\mathbf{C}_3 & \cdots & \mathbf{0} \\ i\mathbf{C}_2 & \mathbf{M}_2 & i\mathbf{C}_3 & \cdots & \cdots & \mathbf{0} \\ \mathbf{M}_2 & i\mathbf{C}_3 & \vdots & \ddots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0,n-1} \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n-1,0} & \Phi_{n-1,1} & \cdots & \Phi_{n-1,n-1} \end{bmatrix} \begin{bmatrix} \Omega_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Omega_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \Omega_{n-1} \end{bmatrix} = \mathbf{0}$$

Orthogonality properties:

$$\sum_{j=1}^{n}\left(\sum_{k=2}^{2j}\boldsymbol{\Omega}^{k-2}\boldsymbol{\Phi}^{\mathrm{T}}i\mathbf{C}_j\boldsymbol{\Phi}\boldsymbol{\Omega}^{2j-k} + \sum_{k=1}^{2j}\boldsymbol{\Omega}^{k-1}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}_j\boldsymbol{\Phi}\boldsymbol{\Omega}^{2j-k}\right) = \mathbf{I}$$

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}_0\boldsymbol{\Phi} + \sum_{j=1}^{n}\left(\sum_{k=1}^{2j-2}\boldsymbol{\Omega}^{k}\boldsymbol{\Phi}^{\mathrm{T}}i\mathbf{C}_j\boldsymbol{\Phi}\boldsymbol{\Omega}^{2j-k-1} + \sum_{k=1}^{2j-1}\boldsymbol{\Omega}^{k}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}_j\boldsymbol{\Phi}\boldsymbol{\Omega}^{2j-k}\right) = \boldsymbol{\Omega}$$

Summary

Equation in the time domain

$$\mathbf{K}_0\mathbf{d} - \sum_{i=1}^{n}(-1)^i \mathbf{M}_i \frac{\partial^{2i}\mathbf{d}}{\partial t^{2i}} = \mathbf{p}(t) \quad (4)$$

e. g.

$$\left(\mathbf{K}_0\mathbf{d} + \mathbf{M}_1 \frac{\partial^2\mathbf{d}}{\partial t^2} - \mathbf{M}_2 \frac{\partial^4\mathbf{d}}{\partial t^4} + \mathbf{M}_3 \frac{\partial^6\mathbf{d}}{\partial t^6} = \mathbf{p}(t)\right)$$

(Set of coupled, higher-order differential equations)

$$\left(\text{from} \quad \left(\mathbf{K}(\omega) - \omega^2\mathbf{M}(\omega)\right)\varphi = \mathbf{0}\right)$$

Mode superposition :

$$\mathbf{d} = \Phi\eta \quad (15)$$

$$\downarrow$$

$$\Omega^2\eta + \ddot{\eta} = \Phi^{\mathrm{T}}\mathbf{p} \quad (17)$$

(Set of uncoupled, second-order differential equations)

Frequency-domain formulation

$$\sigma_{ji},_j + b_i + \rho k^2 u_i = 0 \qquad \text{where} \qquad k^2 = \omega^2 + 2i\zeta\omega$$

Solution: (Laplace/Fourier transforms or)

<u>**Advanced**</u> mode-superposition technique

Structural dynamics

$$\mathbf{\Omega}^2\left(\boldsymbol{\eta} - \boldsymbol{\eta}^b\right) + \ddot{\boldsymbol{\eta}} - \ddot{\boldsymbol{\eta}}^b = \mathbf{\Phi}^{\mathbf{T}}\left(\mathbf{p} - \mathbf{p}^b\right)$$

Diffusion-type problems

$$\mathbf{\Omega}\left(\boldsymbol{\eta} - \boldsymbol{\eta}^b\right) + \dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\eta}}^b = \mathbf{\Phi}^{\mathbf{T}}\left(\mathbf{p} - \mathbf{p}^b\right)$$

Structural dynamics with viscous damping

$$\mathbf{\Omega}\left(\boldsymbol{\eta} - \boldsymbol{\eta}^b\right) - i\left(\dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\eta}}^b\right) = \mathbf{\Phi}^{\mathbf{T}}\left(\mathbf{p} - \mathbf{p}^b\right)$$

Initial displacements and velocities

$$\begin{cases} \boldsymbol{\eta}_{el} = \left[\boldsymbol{\Phi}_{el}^{\mathbf{T}}\mathbf{K}_0\boldsymbol{\Phi}_{el}\right]^{-1}\boldsymbol{\Phi}_{el}^{\mathbf{T}}\mathbf{K}_0\mathbf{d} \\ \boldsymbol{\eta}_{rig} = \boldsymbol{\Phi}_{rig}^{\mathbf{T}}\mathbf{M}_1\mathbf{d} \end{cases}$$

In case of viscous damping:

$$\begin{Bmatrix} \mathbf{d} \\ i\dot{\mathbf{d}} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 \\ \boldsymbol{\Phi}_1\boldsymbol{\Omega}_1 & \boldsymbol{\Phi}_2\boldsymbol{\Omega}_2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{Bmatrix}$$

$$\begin{bmatrix} \boldsymbol{\Phi}_1^{\mathbf{T}}\mathbf{K}_0\boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_1^{\mathbf{T}}\mathbf{K}_0\boldsymbol{\Phi}_2 \\ \boldsymbol{\Phi}_1^{\mathbf{T}}\mathbf{K}_0\boldsymbol{\Phi}_1\boldsymbol{\Omega}_1 & \boldsymbol{\Phi}_1^{\mathbf{T}}\mathbf{K}_0\boldsymbol{\Phi}_2\boldsymbol{\Omega}_2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\Phi}_1^{\mathbf{T}}\mathbf{K}_0\mathbf{d} \\ i\boldsymbol{\Phi}_1^{\mathbf{T}}\mathbf{K}_0\dot{\mathbf{d}} \end{Bmatrix}$$

EVALUATION OF RESULTS AT INTERNAL POINTS

$$\mathbf{p}^*(\omega) = \mathbf{S}(\omega)\left[\mathbf{d}(\omega) - \mathbf{d}^b(\omega)\right]$$ in which $$\mathbf{S}(\omega) = \mathbf{F}^{-1}(\omega)\,\mathbf{H}(\omega)$$

replaced by

$$\mathbf{p}^*(\omega) \approx \sum_{i=0}^{n} \omega^{2i}\,\mathbf{S}_i\left[\mathbf{d}(\omega) - \mathbf{d}^b(\omega)\right]$$ with $$\sum_{i=0}^{n} \omega^{2i}\,\mathbf{S}_i = \left(\sum_{i=0}^{n} \omega^{2i}\,\mathbf{F}_i\right)^{-1} \sum_{i=0}^{n} \omega^{2i}\,\mathbf{H}_i$$

$$\mathbf{p}^*(t) = \sum_{i=0}^{n} \mathbf{S}_i \mathbf{\Phi}\mathbf{\Omega}^{2i}\left(\boldsymbol{\eta} - \boldsymbol{\eta}^b\right)$$

EVALUATION OF RESULTS AT INTERNAL POINTS

$$\mathbf{p}^*(t) = \sum_{i=0}^{n} \mathbf{S}_i \mathbf{\Phi\Omega}^{2i}\left(\boldsymbol{\eta} - \boldsymbol{\eta}^b\right)$$

$$\mathbf{u}(t) = \sum_{j=0}^{m}\sum_{i=0}^{n} \omega_j^{2i}\mathbf{u}_i^*\mathbf{p}^* \equiv \sum_{i=0}^{n} \mathbf{u}_i^*\mathbf{\Omega}^{2i}\mathbf{p}^*$$

$$\Downarrow$$

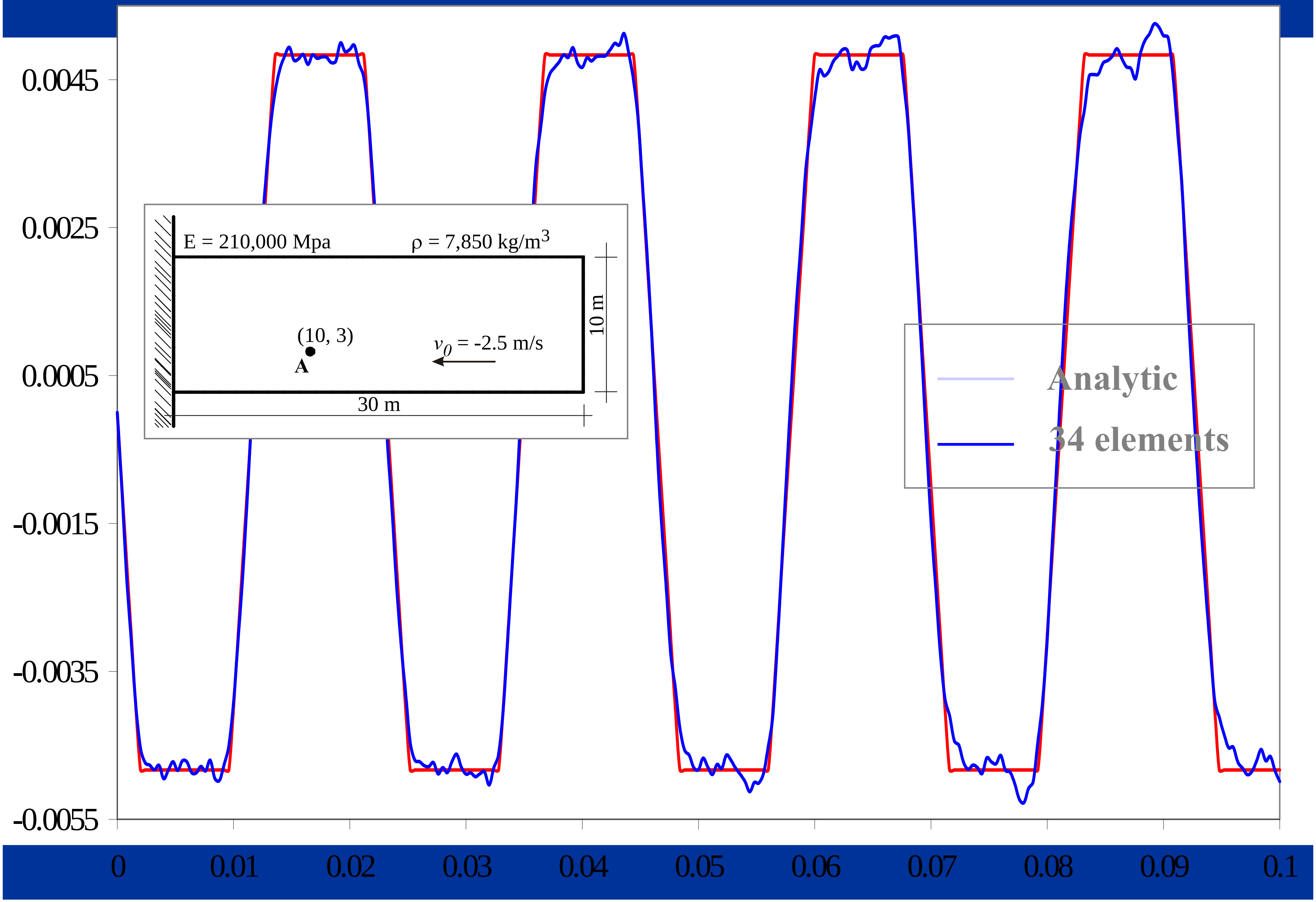
$$\mathbf{u}(t) = \sum_{i=0}^{n}\sum_{j=0}^{i} \mathbf{u}_j^*\mathbf{S}_{i-j}\mathbf{\Phi\Omega}^{2i}\left(\boldsymbol{\eta} - \boldsymbol{\eta}^b\right)$$

for $n = 3$

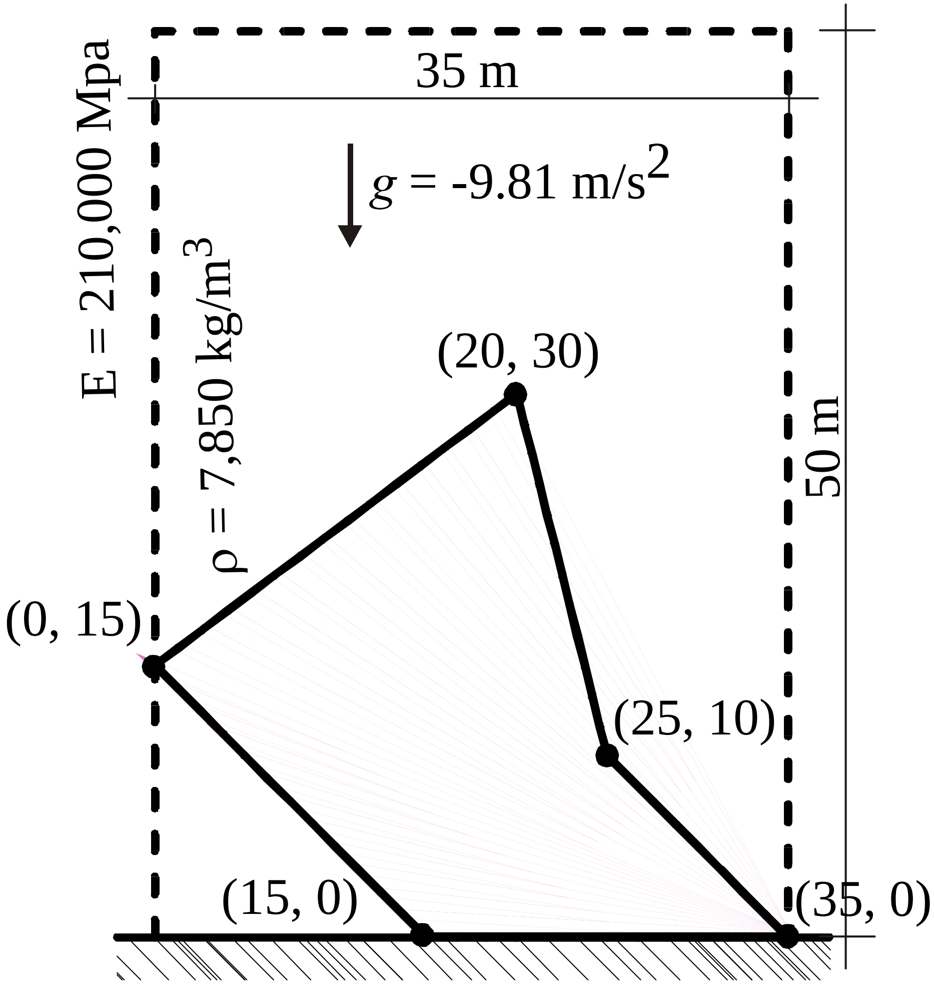
$$\mathbf{u}(t) = \Big[\mathbf{u}_0^*\mathbf{S}_0\mathbf{\Phi} + \left(\mathbf{u}_0^*\mathbf{S}_1 + \mathbf{u}_1^*\mathbf{S}_0\right)\mathbf{\Phi\Omega}^2 + \left(\mathbf{u}_0^*\mathbf{S}_2 + \mathbf{u}_1^*\mathbf{S}_1 + \mathbf{u}_2^*\mathbf{S}_0\right)\mathbf{\Phi\Omega}^4 +$$
$$+ \left(\mathbf{u}_0^*\mathbf{S}_3 + \mathbf{u}_1^*\mathbf{S}_2 + \mathbf{u}_2^*\mathbf{S}_1 + \mathbf{u}_3^*\mathbf{S}_0\right)\mathbf{\Phi\Omega}^6\Big]\boldsymbol{\eta}$$

($\mathbf{S}$ is required for $\mathbf{p}^* = \mathbf{F}^{-1}\mathbf{Hd} \equiv \mathbf{Sd}$)

Example 1: Initial velocity – Displacement at point A



Transient Analysis – Example 2: Gravitational force



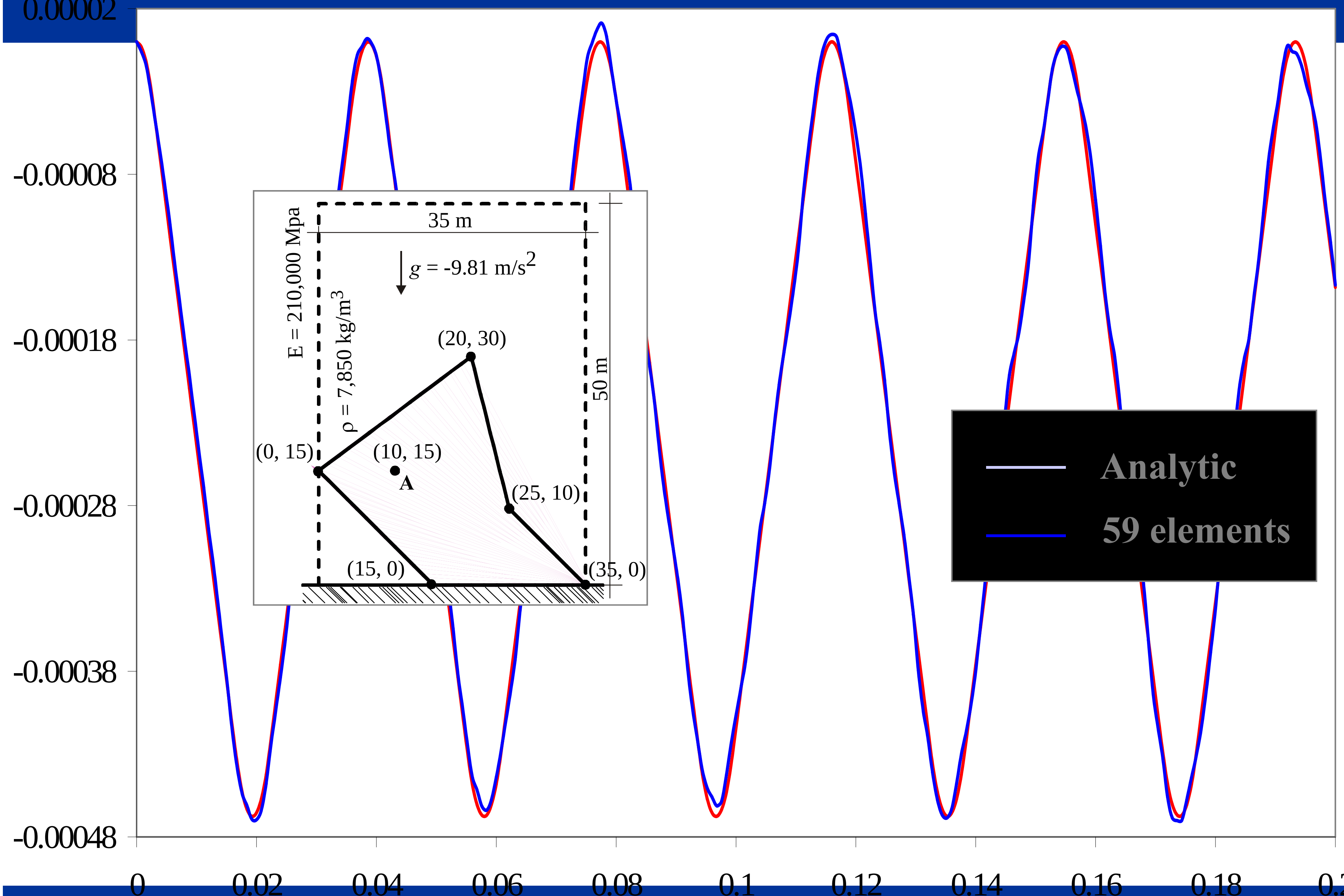
59 equally spaced,
linear boundary elements

$$u = \sum_{n=1}^{\infty} \frac{4g\ \sin(k_n\, x)\ [1 - \cos(\omega_n\, t)]}{(2n-1)\ \pi\ \omega_n^2}$$

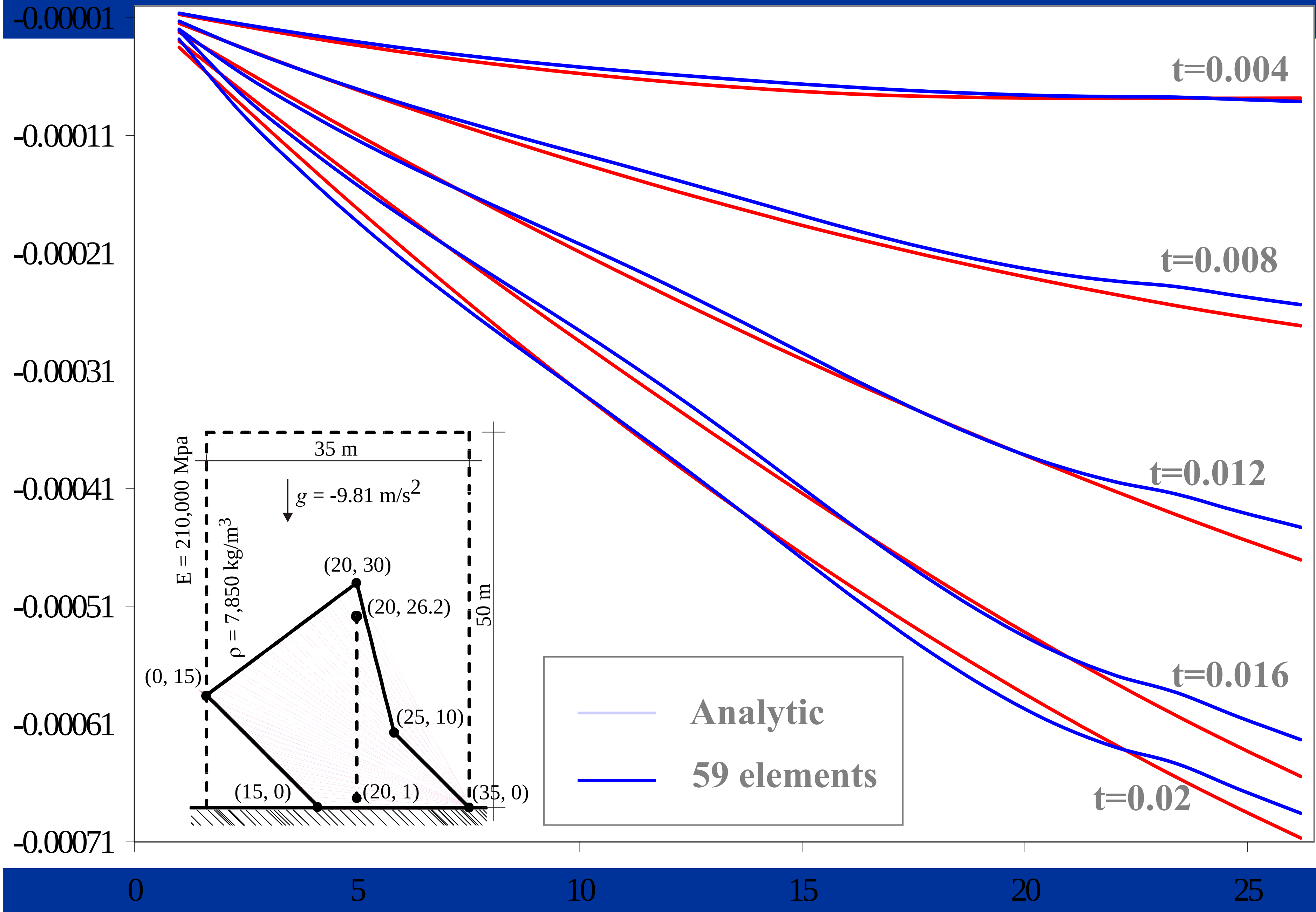
$$\begin{cases} \omega_n = k_n \sqrt{E/\rho} \\ k_n = \dfrac{(2n-1)\pi}{2L} \end{cases}$$

$\upsilon = 0$ (can be solved in the frame of the theory of potential)

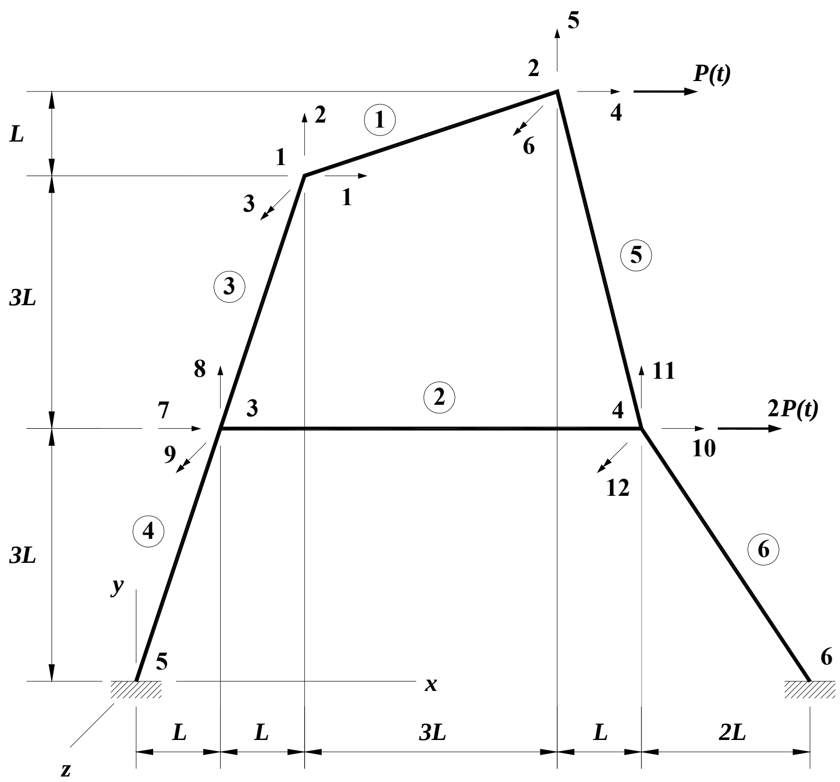
Example 2: Gravitational force – Displacement at point A

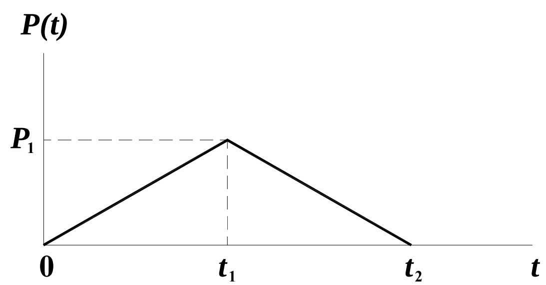


Example 2: Gravitational force – Displacements along a line segment



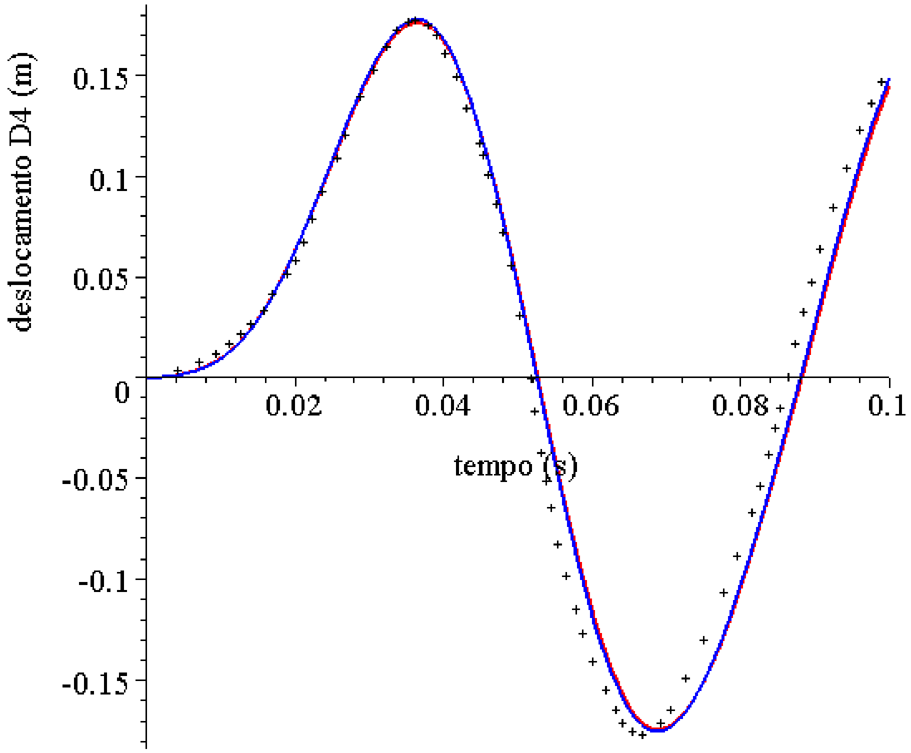
Plane frame with 6 members and 12 d.o.f.

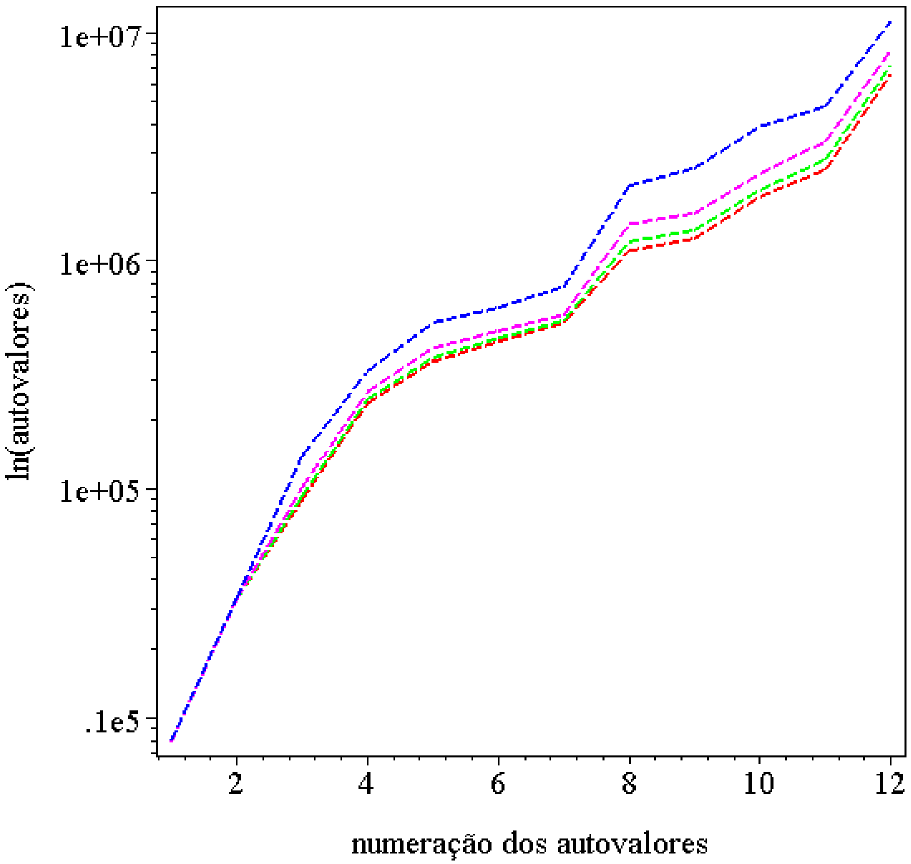




(Weaver, W., Jr., and Johnston, P. R., 1987. *Structural Dynamics by Finite Elements*, Prentice-Hall Inc., Englewood Cliffs, New Jersey)

Plane frame with 6 members and 12 d.o.f.





Displacement results with 1 and 4 mass matrices, as compared with values given by Weaver and Johnston.

Eigenvalues obtained for the problem with 1, 2, 3 and 4 mass matrices.

Contents of this presentation

- Variationally-based, hybrid boundary and finite elements
- Developments for time-dependent problems
- Developments in gradient elasticity
- Dislocation-based formulations (for fracture mechanics)
- From the collocation (conventional or hybrid) boundary element method to a meshless formulation

  Or: The expedite boundary element method

A Family of 2D and 3D Hybrid Finite Elements for Strain Gradient Elasticity

Ney Augusto Dumont

Daniel Huamán Mosqueira

e-mail: dumont@puc-rio.br

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CAPES

Conclusions (1)

$$\sigma_{ji} = \tau_{ji} - g^2 \tau_{ji,kk}$$ Total stress

- The hybrid BEM / FEM (a two-field formulation) is a natural variational tool to deal with gradient elasticity
- Singular and nonsingular fundamental solutions were either redeveloped for BEM or originally developed for FEM gradient elasticity implementations
- General families of finite elements were obtained
- Extension to time-dependent problems in the frequency domain is straightforward (in progress – already done for truss and beam elements)
- Simple implementations for truss and beam elements: was a fruitful apprenticeship (disagreement with some results in the literature - symmetry and representation of constant strain state)
- **Treatment of the normal displacement gradient on Γ is different from Mindlin's proposition** (our results still remain to be validated)
- Numerical examples for 2D problems are being implemented

Conclusions (2)

$$\sigma_{ji} = \tau_{ji} - g^2 \tau_{ji,kk}$$ Total stress

- .
- .
- .
- **Treatment of the normal displacement gradient on Γ is different from Mindlin's proposition** (our results still remain to be validated)
- Numerical examples for 2D problems are being implemented:
  - Orthogonality to rigid body displacements: OK!
  - Symmetry of the flexibility matrix **F***: OK!
  - Patch tests for linear displacements fields: OK!
  - Patch tests for non-linear displacement fields: convergence still questionable.

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Conclusions

Expedite formulation of the boundary element method (in progress)

- **Implementation extremely advantageous for:**
  - **large problems**
  - **problems with complicated fundamental solutions (time-dependent, axisymmetric, for gradient elasticity, etc)**
- **Numerical accuracy still under investigation, but comparable to the conventional BEM**
- **Evaluation of results at internal points requires no further integrations**
- **However, results close to the boundary require the knowledge of the null space V (which may be obtained via Gauss-Seidel iteration in principle with a simple pre-conditioning of H^T)**

Work in progress

- **Implementation in Fortran for large 2D potential and elasticity problems in the frequency domain**
- **Implementation for strain gradient elasticity**