POLITECNICO DI MILANO

NSF Workshop on the Emerging Applications and Future Directions of the Boundary Element Method

September 1-3, 2010

Cleveland, Ohio, USA (The University of Akron Campus)

BEM for modeling (dissipation) in microelectromechanical systems (MEMS)

Attilio Frangi *attilio.frangi@polimi.it*

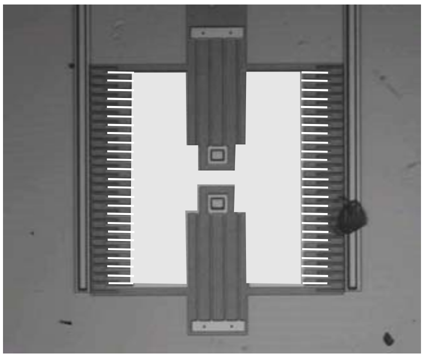
Department of Structural Engineering, Politecnico di Milano

Department of Mechanics, Ecole Polytechnique, Paris

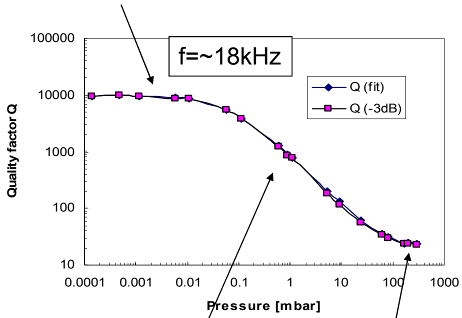
INTRODUCTION

DIFFERENT SOURCES OF DISSIPATION

Example: Tang resonator



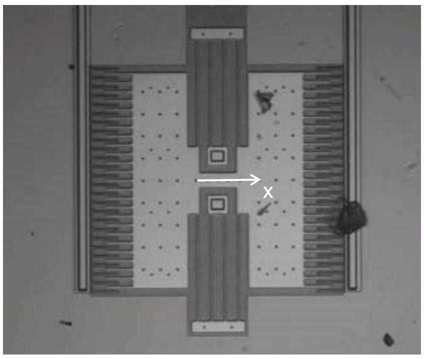
solid/surface damping: internal friction, thermoelastic damping



fluid damping: rarefied regime

fluid damping: continuum regime

TYPICAL APPLICATION: reduced parameter model for Q extraction

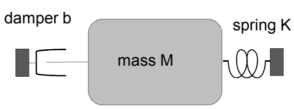


Usual assumptions:

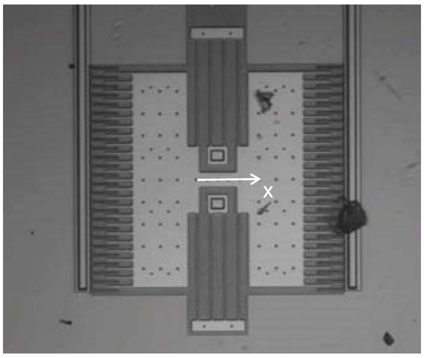
•only springs can deform but contribution to dissipation is negligible; shuttle is *rigid* (displacement denoted by *U(t)* w.r.t. reference to rest position)

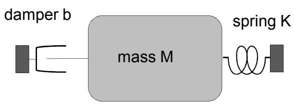
•*small perturbations* (*linear* response)

•low resonating frequencies imply *instantaneous* fluid response



TYPICAL APPLICATION: reduced parameter model for Q extraction





Usual assumptions:

•only springs can deform but contribution to dissipation is negligible; shuttle is *rigid* (displacement denoted by *U(t)* w.r.t. reference to rest position)

•*small perturbations* (*linear* response)

•low resonating frequencies imply *instantaneous* fluid response

As a consequence a quasi-static approach applies and the force exerted by the gas on the structure has the form:

$$\mathbf{T}(\mathbf{x},t) = \mathbf{t}(\mathbf{x})\dot{U}(t)$$

and $\mathbf{t}(\mathbf{x})$ is a real vector function of position.
All is needed is a tool to estimate $\mathbf{t}(\mathbf{x})$ when a *unit velocity is imposed to the shuttle*

Overall gas action on structure along direction x is:

$$F(t) = \left(\int_{\partial\Omega} t_x(\mathbf{x})\, dS\right)\dot{U}(t)$$

Eventually:

$$B = \int_{\partial\Omega} t_x(\mathbf{x})\, dS$$

$$\boxed{M\ddot{U}(t) + B\dot{U}(t) + KU(t) = F(t)}$$

REDUCED PARAMETER MODEL

$$M\ddot{U}(t) + B\dot{U}(t) + KU(t) = F(t)$$

$$\Downarrow$$

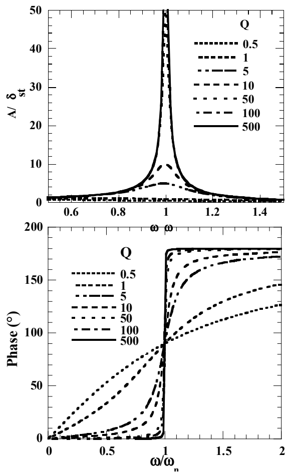
$$\ddot{U}(t) + 2\xi\omega_0\dot{U}(t) + \omega_0^2 U(t) = \frac{F(t)}{M}$$

Rigorous definition of Q:

$$\boxed{Q = \frac{1}{2\xi} = \omega_0 \frac{M}{B}}$$

These assumptions lead to a quality factor *Q* which depends on pressure only through *B*

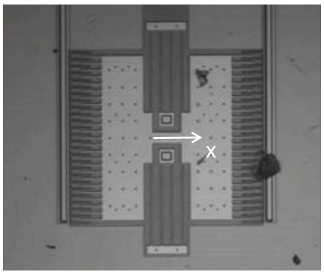
Q crucial in gyroscopes, magnetometers, etc…



Assumptions can be validated with phase or amplitude diagrams obtained experimentally

Impose a sinusoidal input, measure output and phase shift between input and output

REDUCED PARAMETER MODEL: accounting for deformability



If also springs contribute to gas damping (and more in general when an accurate estimate of *M* is required accounting also for spring contribution), one typically assumes:

$$u(\mathbf{x},t) = g(\mathbf{x})U(t)$$

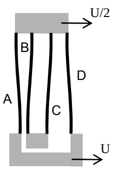
where g(**x**) is a typical mode of the structure (static or dynamic) and reduces to a constant on rigid shuttle.
Then the Principle of Virtual Power is enforced using as virtual velocity field:

$$\tilde{u}(\mathbf{x}) = g(\mathbf{x})\tilde{U}$$

$$\int_{\Omega} \tilde{u}(\mathbf{x})\rho\ddot{u}(\mathbf{x},t)d\Omega + \int_{\Omega} \boldsymbol{\varepsilon}[\tilde{u}](\mathbf{x}) : \boldsymbol{\sigma}[u](\mathbf{x},t)d\Omega - \int_{\partial\Omega} \tilde{u}(\mathbf{x})T_x(\mathbf{x},t)dS + ... = 0$$

PVP

$$\mathbf{T}(\mathbf{x},t) = \mathbf{t}(\mathbf{x})\dot{U}(t)$$



$$u_B(x) = u_C(x) = \frac{U}{2} f(x)$$

$$u_A(x) = u_D(x) = \frac{U}{2}(1 + f(x))$$

$$f(x) = \left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right)$$

deformed configuration

If **t**(**x**) denotes the force on the structure when the velocity g(**x**) is enforced, then the virtual power of viscous forces is:

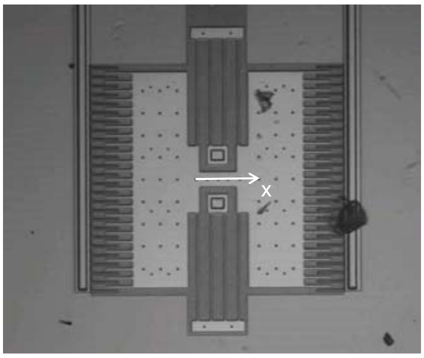
$$\mathbb{P}_V(\tilde{U},U) = \tilde{U}\left(\int_{\partial\Omega} g(\mathbf{x})t_x(\mathbf{x})\,dS\right)\dot{U}(t)$$

eventually yielding:

$$B = \int_{\partial\Omega} g(\mathbf{x})t_x(\mathbf{x})\,dS$$

$$\boxed{M\ddot{U}(t) + B\dot{U}(t) + KU(t) = F(t)}$$

REDUCED PARAMETER MODEL: accounting for higher frequencies



If frequencies increase but linearity is preserved (almost always true the case of inertial MEMS), than one has to work in the frequency domain

$$u(\mathbf{x},t) = g(\mathbf{x})Ue^{i\omega t}$$

If **T**(**x**,t) denotes the force exerted on the structure by the fluid, then typically:

$$\mathbf{T}(\mathbf{x},t) = \mathbf{t}(\mathbf{x},\omega)i\omega Ue^{i\omega t}$$

with **t**(**x**,ω) $\in \mathbb{C}$ hence, introducing the complex damping coefficient:

$$B(\omega) = \int_{\partial\Omega} g(\mathbf{x})t_x(\mathbf{x},\omega)\,dS$$

the final 1D model writes:

$$\boxed{(-\omega^2 M + i\omega B(\omega) + K)U = F}$$

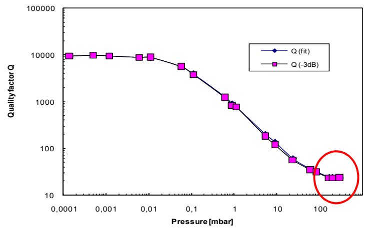
with *B* contributing to both damping and stiffness

KNUDSEN NUMBER AND FLOW MODELS

Knudsen number Kn = λ/L

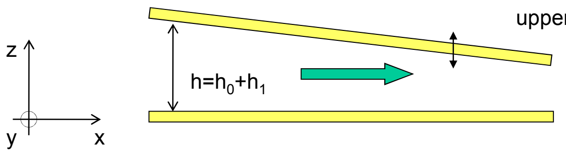
λ: mean free path of molecules
L: characteristic length scale

λ=0.069 µm at SATP, λ~1/p



- **Navier-Stokes equations (no-slip):** **$Kn \leq 10^{-3}$**
- **Navier-Stokes equations (slip):** **$10^{-3} \leq Kn \leq 10^{-1}$**
- **Transition regime:** **$10^{-1} \leq Kn \leq 10$**
- **Free molecular flow:** **$Kn > 10$**

Simplified models: linearized Reynolds equation



upper plate moving in vertical direction

fixed plate

typical of lubrification theory

obtained by simply imposing the mass balance equation where q_x is the total flux along x and ρ is density (indep. of z) ⇨

$$\frac{\partial \rho h}{\partial t} + \frac{\partial q_x}{\partial x} = 0$$

and inserting for q_x the flux obtained from the Poiseuille parabolic velocity distribution

then linearise w.r.t. to pressure: $p=p_0+p_1$ ⇨

$$\frac{h_0^2}{12\eta} p_0 \frac{\partial^2}{\partial x^2}\left(\frac{p_1}{p_0}\right) - \frac{\partial}{\partial t}\left(\frac{p_1}{p_0}\right) = \frac{\partial}{\partial t}\left(\frac{h_1}{h_0}\right)$$

- h_0 small w.r.t. plate dimensions L
- variations h_1 of h_0 small w.r.t. to h_0
- isothermal process

if this term is neglected eq. is called incompressible Reynolds

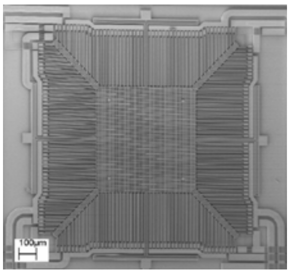
various extensions to account for edge effects of rectangular and circular plates, holed plates, etc.

Blech, J. J., On Isothermal Squeeze Films, *J. Lubrication Technology,* 105, 1983.
Bao M., *Yang H.*, Squeeze film air damping in MEMS, *Sensors and Actuators A* ,136, 3–27, 2007 (Review)

Full Navier-Stokes model and simplifications

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \eta\Delta\mathbf{u} \qquad \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$$

- Reynolds number Re = UL/ν (U typical speed, L typical dimension, ν kinematic viscosity)
  if Re << 1 neglect non-linear convective terms in Navier Stokes
- Mach number M = U/c (U typical speed, c speed of sound)
  if M << 1 set $\nabla\cdot u = 0$ (incompressibility) in Navier-Stokes
- Stokes number St = fL^2/ν (f vibration frequency)
  if St << 1 neglect inertia terms in Navier-Stokes



Example of biaxial accelerometer (SI units):
L ~ $2.6{\times}10^{-6}$ m; f = 4400Hz; ν =1.5 10^{-5}; U ~ 2πfD ;
D < 1/10 L (D amplitude of oscillation)
U ~ $7{\times}10^{-3}$ Re ~ 10^{-3} M ~ $7{\times}10^{-4}$ St ~ $7{\times}10^{-3}$

⇩

Incompressible (quasi-static) Stokes formulation

QUASI STATIC STOKES PROBLEM

$$\nabla p(\mathbf{x}) - \eta\Delta\mathbf{u}(\mathbf{x}) = \mathbf{0} \qquad \nabla\cdot\mathbf{u}(\mathbf{x}) = 0 \qquad \text{in } \Omega_\infty\text{-}\Omega$$
$$\mathbf{u}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) - c_t\mathbf{t}^S(\mathbf{x}) \qquad \text{on } S$$

p pressure (defined up to a constant!)
u fluid velocity, **g** structure velocity
t = **σ**.**n** tractions, **t**^{S} tractions projected on surface

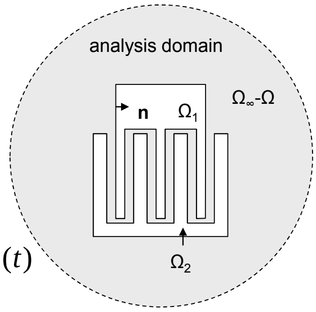
$$\boldsymbol{\sigma}(\mathbf{x}) = -p(\mathbf{x})\mathbb{1} + \eta\left(\nabla\mathbf{u}(\mathbf{x}) + \nabla^T\mathbf{u}(\mathbf{x})\right)$$

Basic assumption:
fluid response to structure motion is instantaneous.

Time dependence?

1. inertia terms are dropped
2. velocity of structures is enforced as boundary conditions

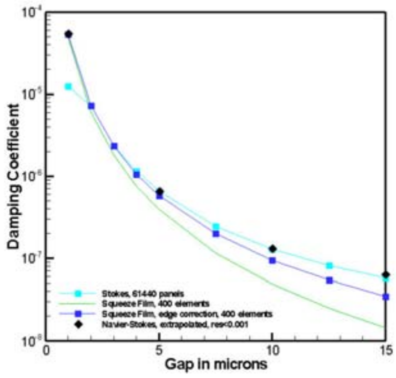
$$\dot{S}(\mathbf{x},t) = g(\mathbf{x})\dot{U}(t)$$



STATIC STOKES PROBLEM WITH DIRICHLET BC
formally identical to incompressible elasticity

Comparison between full models and Reynolds

Model problem to compare Navier-Stokes, Stokes, and Reynolds solutions for damping coefficients. A 50x50x4 micron plate in motion above a 60x60x4 micron plate at 1atm, 300K.



Comparison of Stokes, Reynolds Navier-Stokes, solutions

STOKES PROBLEM BY BEM: MVT and slip BC

$$\nabla p(\mathbf{x}) - \eta \Delta \mathbf{u}(\mathbf{x}) = \mathbf{0} \qquad \nabla \cdot \mathbf{u}(\mathbf{x}) = 0 \qquad \text{in } \Omega$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) - c_t \mathbf{t}^S(\mathbf{x}) \qquad \text{on } S$$

t tractions, **t**^{S} tractions projected on surface

$$c_t := \frac{2-\sigma}{\sigma}\frac{\lambda}{\eta}$$

$$\mathbf{t}^S(\mathbf{x}) = [\mathbb{1} - \mathbf{n}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{x})] \cdot \mathbf{t}(\mathbf{x})$$

$$\boxed{\begin{aligned} \mathbf{g}(\mathbf{x}) - \frac{c_t}{2}\mathbf{t}^S(\mathbf{x}) - \frac{\gamma}{\eta}\frac{1}{2}\mathbf{t}(\mathbf{x}) = \int_S \Big\{ & \boldsymbol{\mathcal{V}}(\mathbf{r}) \cdot \mathbf{t}(\mathbf{y}) + c_t\, [\boldsymbol{\mathcal{K}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{y})] \cdot \mathbf{t}^S(\mathbf{y}) \\ & + \frac{\gamma}{\eta} \left([\mathbf{n}(\mathbf{x}) \cdot \boldsymbol{\mathcal{K}}(\mathbf{r})] \cdot \mathbf{t}(\mathbf{y}) - c_t\, [\mathbf{n}(\mathbf{x}) \cdot \boldsymbol{\mathcal{W}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{y})] \cdot \mathbf{t}^S(\mathbf{y}) \right) \Big\} \mathrm{d}S_y \end{aligned}}$$

MVT: Mixed Velocity Traction. The two equations are linearly combined

STOKES PROBLEM BY BEM: MVT

$$\mathbf{g}(\mathbf{x}) - \frac{c_l}{2}\mathbf{t}^S(\mathbf{x}) = \int_S \{\mathscr{V}(\mathbf{r}) \cdot \mathbf{t}(\mathbf{y}) + c_l[\mathscr{K}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{y})] \cdot \mathbf{t}^S(\mathbf{y})\}\, \mathrm{d}S_y$$

velocity equation

$$V_{ik}(\mathbf{r}) = \frac{1}{8\pi\eta}\left(\frac{\delta_{ik}}{r} + \frac{r_i r_k}{r^3}\right) \qquad K_{iqk}(\mathbf{r}) = -\frac{3}{4\pi}\frac{1}{r^5} r_i r_q r_k$$

traction equation

$$\frac{1}{2}\mathbf{t}(\mathbf{x}) = \int_S \{-[\mathbf{n}(\mathbf{x}) \cdot \mathscr{K}(\mathbf{r})] \cdot \mathbf{t}(\mathbf{y}) + c_l[\mathbf{n}(\mathbf{x}) \cdot \mathscr{W}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{y})] \cdot \mathbf{t}^S(\mathbf{y})\}\, \mathrm{d}S_y$$

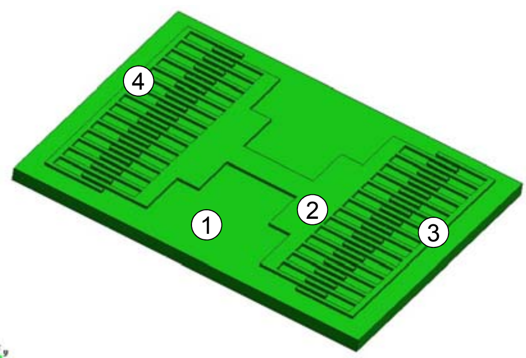
$$W_{qiks}(\mathbf{r}) = \frac{\eta}{4\pi}\frac{1}{r^3}\left[2\delta_{sk}\delta_{iq} + \frac{3}{r^2}(\delta_{ik}r_q r_s + \delta_{kq}r_i r_s + \delta_{is}r_k r_q + \delta_{sq}r_i r_k) - 30\frac{r_i r_k r_q r_s}{r^4}\right]$$

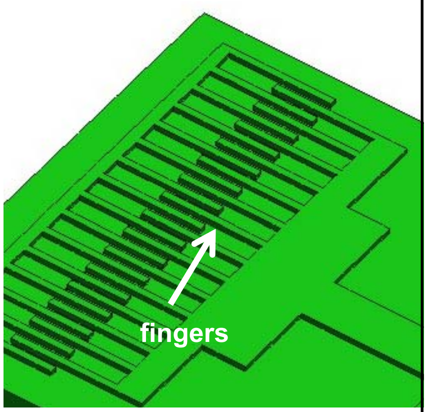
very same kernels as in incompressible elsticity!!

NULL SPACE OF VELOCITY EQUATION

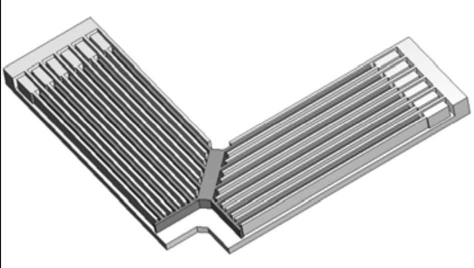
Exact null space vs numerical null space

$$\mathbf{t}^\alpha(\mathbf{x}) = \begin{cases} \mathbf{n}(\mathbf{x}), & \mathbf{x} \in S^\alpha \\ 0 & \text{elsewhere} \end{cases}, \qquad 1 \leq \alpha \leq N$$





WHY BEM?



Largest problem dimension on a desktop

N_{BEM}: $\sim \alpha\, 10^6$

Equivalent FEM problem dimension

N_{FEM}: $\sim N_{BEM}^{3/2} \sim \beta\, 10^9$

Infinite extent of air domain

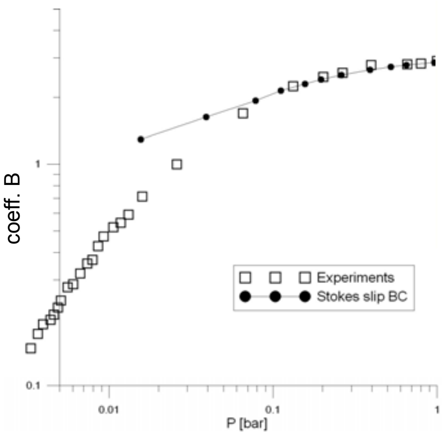
- No mesh of air domain
- No unknowns in air domain

Tractions are primal variables hence very good accuracy

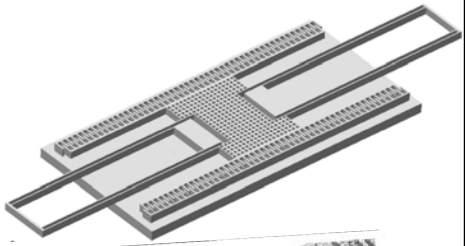
•GMRES iterative solver
•Classical FMM with adpative octree
•Truncation order p ~ 8
•Computation on the fly of near integrals
•Block diagonal preconditioning
•OpenMP parallelisation

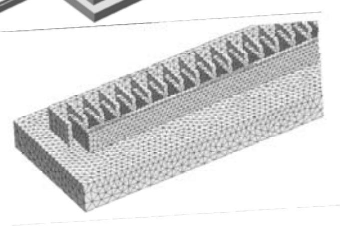
Frangi A., Di Gioia A., Multipole BEM for the evaluation of damping forces on MEMS, *Computational Mechanics*, 37, 24-31 (2005)
Frangi A, Tausch J., A qualocation enhanced approach for the Dirichlet problem of exterior Stokes flow, *Eng. Anal. Boundary Elem.*, 29, 86–93 (2005)
Frangi A, Spinola G., Vigna B. : On the evaluation of damping in MEMS in the slip-flow regime, *Int. J. Num. Meth. Engng.*, 68, 1031–1051 (2006)

TANG RESONATOR



diverge at low pressure since continuum model fails

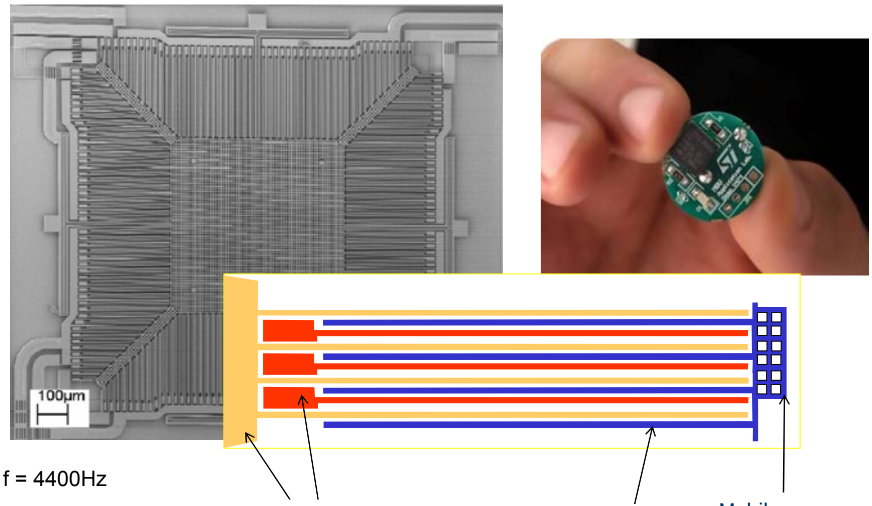




$$B = \int_{\partial\Omega} t_x(\mathbf{x})\, dS$$

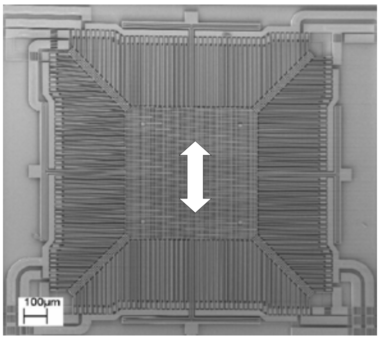
$$M\ddot{U}(t) + B\dot{U}(t) + KU(t) = F(t)$$

BIAXIAL ACCELEROMETER



f = 4400Hz

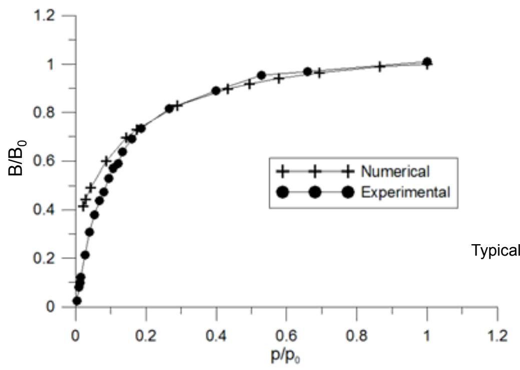
EXPERIMENTAL VALIDATION: LINEAR ACCELEROMETER AND SLIP B.C.



p = 1 bar

	Numerical "no slip"	Numerical "slip"	Experimental
Squeeze flow	$2.32\,10^{-4}$ N	$2.10\,10^{-4}$ N	-
Couette flow	$7.37\,10^{-6}$ N	$7.03\,10^{-6}$ N	-
Mass with holes	$2.10\,10^{-6}$ N	$1.94\,10^{-6}$ N	-
Total force	$2.41\,10^{-4}$ N	$2.19\,10^{-4}$ N	$2.21\,10^{-4}$ N

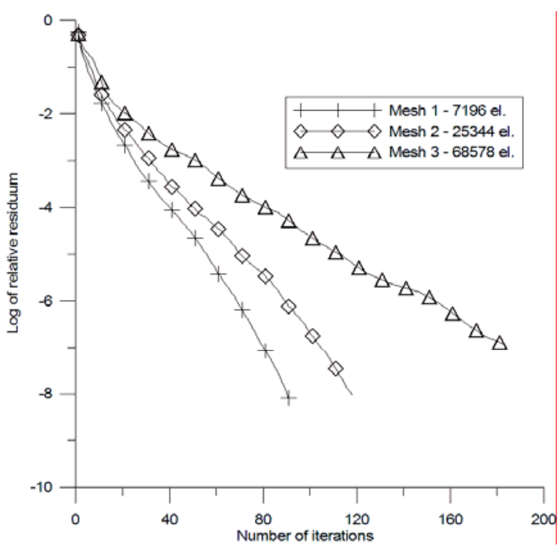
EXPERIMENTAL VALIDATION: LINEAR ACCELEROMETER AND SLIP B.C.

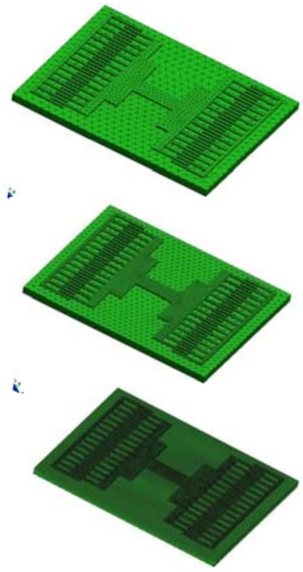


Typical comparison of simulations and results

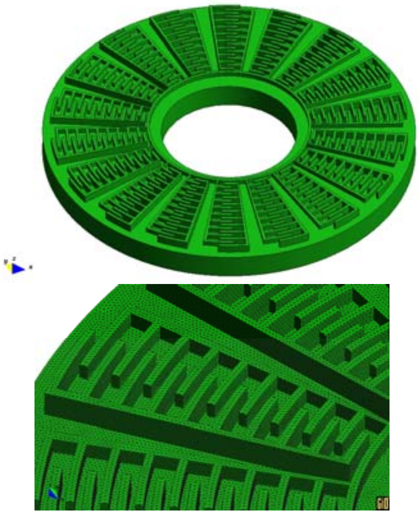
excellent agreement at high pressure
diverge at low pressure
since continuum model fails

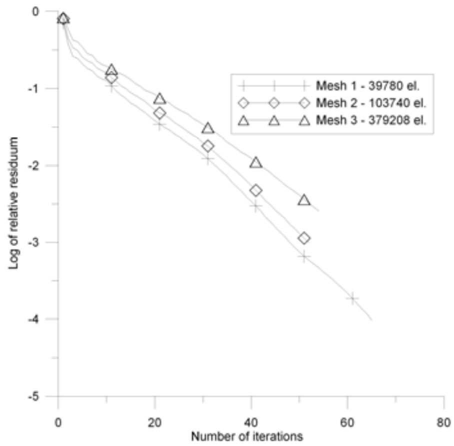
Examples of full scale analysis: comb finger resonator



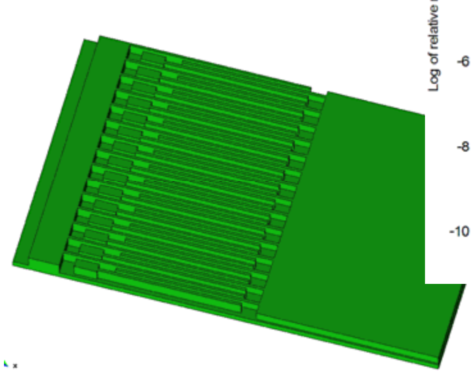


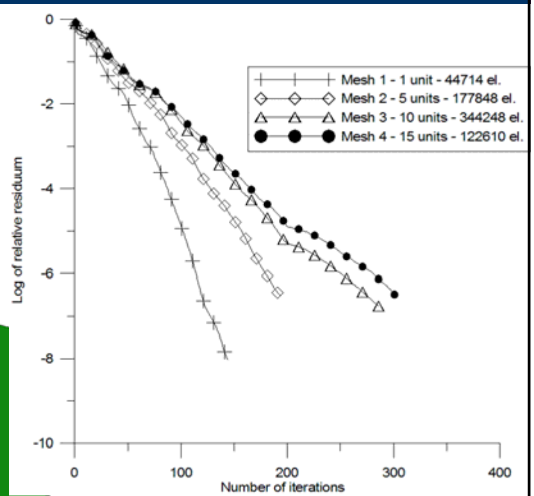
Examples of full scale analysis: rotational resonator





Examples of full scale analysis: parallel plates resonator





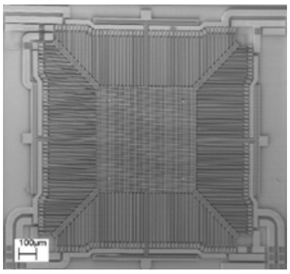
EXTENSION TO HIGHER WORKING FREQUENCIES

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \eta\Delta\mathbf{u} \qquad \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$$

Reynolds number Re = UL/ν (U typical speed, L typical dimension, ν kinematic viscosity)
if Re << 1 neglect non-linear convective terms in Navier Stokes

Mach number M = U/c (U typical speed, c speed of sound)
if M << 1 set ∇·u = 0 (incompresibility) in Navier-Stokes

Stokes number St = fL^2/ν (f vibration frequency)
if St << 1 neglect inertia terms in Navier-Stokes



Example of biaxial accelerometer (SI units):
L ~ 2.6 $\times 10^{-6}$ m; f = 4400Hz; ν =1.5 10^{-5}; U ~ 2πfD ;
D < 1/10 L (D amplitide of oscillation)
U ~ 7$\times 10^{-3}$ Re ~ 10^{-3} M ~ 7$\times 10^{-4}$ St ~ 7$\times 10^{-3}$

⇩

Incompressible frequency-domain Stokes formulation

INCOMPRESSIBLE OSCILLATORY STOKES FLOW

Mixed Velocity Traction Formulation

$$\boxed{\mathbf{g}(\mathbf{x}) - \frac{\gamma}{\eta}\frac{1}{2}\mathbf{t}(\mathbf{x}) = \int_S \left\{ \boldsymbol{\mathcal{V}}(\mathbf{r})\cdot\mathbf{t}(\mathbf{y}) + \frac{\gamma}{\eta}\left[\mathbf{n}(\mathbf{x})\cdot\boldsymbol{\mathcal{K}}(\mathbf{r})\right]\cdot\mathbf{t}(\mathbf{y}) \right\} \mathrm{d}S_y}$$

$$V_{ik}(\mathbf{r}) = \frac{1}{8\pi\mu}A(R)\frac{\delta_{ik}}{r} + B(R)\frac{r_i r_k}{r^3}$$

Kernels involved

$$K_{iqk}(\mathbf{r}) = -\frac{\delta_{ij}r_k + \delta_{kj}r_i}{4\pi r^3}\left[e^{-R}(R+1) - B\right] - \frac{\delta_{ik}r_j(1-B)}{4\pi r^3} - \frac{r_i r_j r_k}{4\pi r^5}\left[5B - 2e^{-R}(R+1)\right]$$

$$A = 2e^{-R}\left(1 + \frac{1}{R} + \frac{1}{R^2}\right) - \frac{2}{R^2}, \quad B = -2e^{-R}\left(1 + \frac{3}{R} + \frac{3}{R^2}\right) + \frac{6}{R^2},$$

$$R = r\sqrt{\frac{-i\omega\rho}{\eta}} \Rightarrow \boxed{(-1+i)}r\sqrt{\frac{\omega\rho}{2\eta}}$$

if $\max(r)\sqrt{\frac{\omega\rho}{\eta}} \ll 1$ static limit recovered

$$K(\mathbf{r}) = \frac{\exp(ikr)}{4\pi r} \qquad k = (\alpha + \mathrm{i}\beta)\vartheta$$

Wave propagation in lossy media (e.g. soils). Linear constitutive relations involve complex moduli leading to complex-valued wavenumbers with α=±1 0<β<1 (often with β<<1)

Eddy currents for the design of electrical transformers (|α|=β=1)
(Schmindlin et al. 2001)

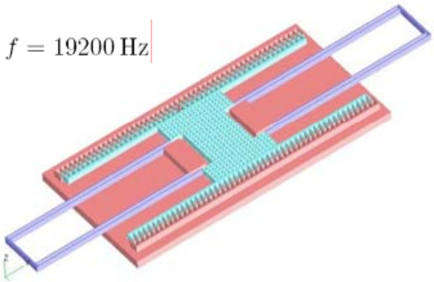
Transient Stokes flow for the analysis of dissipation in Micro-Systems (|α|=β=1)
(Ye et al. 2003)

Computation of Casimir forces (attractive forces arising between uncharged conductive surfaces in vacuum) (α =0, β=1)
(Reid et al. 2009)

Optical tomography with α = -1; β >1.
(Zacharopoulos et a. 2006)

Classical formulation with Gegenbauer addition theorem very effective if β is large enough w.r.t. to α
Frangi and Bonnet, CMES, 2010

$f = 19200\,\mathrm{Hz}$



Finger gap	2.88μm
Finger length	40.05μm
Finger overlap	19.44μm
Center plate	$54.9 \times 19.26\mu\mathrm{m}^2$
Side plate1 $\times 2$	$28.26 \times 89.6\mu\mathrm{m}^2$
Side plate2 $\times 4$	$11.3 \times 40.5\mu\mathrm{m}^2$
Thickness	1.96μm
Substrate gap	2μm
Truss length	78μm
Truss width	13μm

Drag force (pN)	Steady	Unsteady
Bottom	508.75	510.72
Side	284.84	294.50
Top	102.31	142.8
Total	895.9	948.02

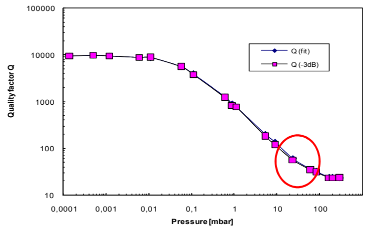
$$L\sqrt{\frac{\omega\rho}{\eta}} \ll 1$$

KNUDSEN NUMBER AND FLOW MODELS

Knudsen number Kn = λ/L

λ: mean free path of molecules
L: characteristic length scale

λ=0.069 μm at SATP, λ~1/p



- **Navier-Stokes equations (no-slip):** **$Kn \leq 10^{-3}$**
- **Navier-Stokes equations (slip):** **$10^{-3} \leq Kn \leq 10^{-1}$**
- **Transition regime:** **$10^{-1} \leq Kn \leq 10$**
- **Free molecular flow:** **$Kn > 10$**

BGK MODEL FOR BOLTZMANN EQUATION

$f(x,\xi)$: mass density probability depends on location x and molecular velocity ξ

$$\rho = \int_{\mathbb{R}^3} f \, \mathrm{d}\boldsymbol{\xi}$$

$$\rho \mathbf{v} = \int_{\mathbb{R}^3} f \boldsymbol{\xi} \, \mathrm{d}\boldsymbol{\xi}$$

$$T = \frac{1}{3\mathcal{R}\rho} \int_{\mathbb{R}^3} f \, |\boldsymbol{\xi} - \mathbf{v}|^2 \, \mathrm{d}\boldsymbol{\xi}$$

$$\boldsymbol{\sigma} = \int_{\mathbb{R}^3} f \, (\boldsymbol{\xi} - \boldsymbol{v}) \otimes (\boldsymbol{\xi} - \boldsymbol{v}) \, d\boldsymbol{\xi}$$

mass density probability for a gas at rest

$$f_0 = \frac{\rho_0}{(2\pi \mathcal{R} T_0)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{2\mathcal{R}T_0}\right)$$

rest Maxwellian

BGK model

$$\boxed{\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla f = \overbrace{\nu(\rho, T)(f_M - f)}^{\text{rhs accounting for molecule collisions}}}$$

$$\boxed{f_M = \frac{\rho}{(2\pi \mathcal{R} T)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi} - \mathbf{v}|^2}{2\mathcal{R}T}\right)}$$

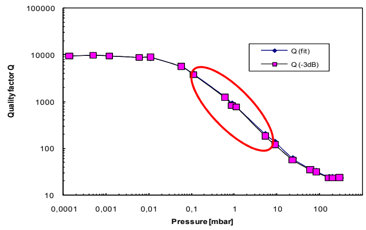
local Maxwellian

KNUDSEN NUMBER AND FLOW MODELS

Knudsen number $Kn = \lambda/L$

λ: mean free path of molecules
L: characteristic length scale

λ=0.069 μm at SATP, $\lambda \sim 1/p$



- **Navier-Stokes equations (no-slip):** **$Kn \leq 10^{-3}$**
- **Navier-Stokes equations (slip):** **$10^{-3} \leq Kn \leq 10^{-1}$**
- **Transition regime:** **$10^{-1} \leq Kn \leq 10$**
- **Free molecular flow:** **$Kn > 10$**

FREE MOLECULE FLOW

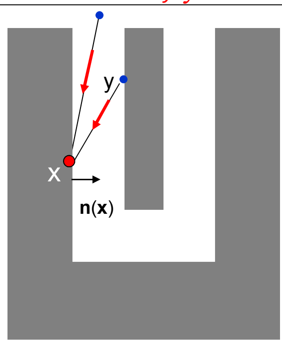
$f(x,\xi)$: mass density probability depends on location x and molecular velocity ξ

$$\boxed{\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla f = 0}$$

collisions between molecules neglected

for molecules coming from other MEMS surfaces:

$$f(\mathbf{x}, \boldsymbol{\xi}, t) = f(\mathbf{y}, \boldsymbol{\xi}, t - \frac{r}{\xi}) \qquad \mathbf{y} = \mathbf{x} + \mathbf{r}, \quad \mathbf{r} = -r\frac{\boldsymbol{\xi}}{\xi}$$



for molecules coming from far field region:

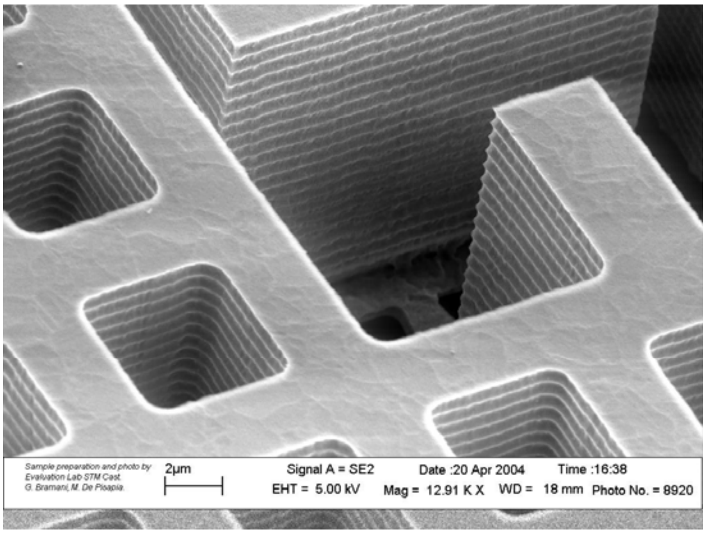
$$f(\mathbf{x}, \boldsymbol{\xi}) = f_0(\boldsymbol{\xi}) = \frac{\rho_0}{(2\pi \mathcal{R} T_0)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{2\mathcal{R} T_0}\right)$$

Diffuse model for molecules re-emitted from surfaces

$$f(\mathbf{x}, \boldsymbol{\xi}) = \frac{\rho_w(\mathbf{x})}{(2\pi \mathcal{R} T_w(\mathbf{x}))^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi} - \mathbf{w}(\mathbf{x})|^2}{2\mathcal{R} T_w(\mathbf{x})}\right) \quad \text{for} \quad (\boldsymbol{\xi} - \mathbf{w}) \cdot \mathbf{n} > 0$$

incoming flux of molecules $\rho_w(\mathbf{x}) = \left(\frac{2\pi}{\mathcal{R} T_w(\mathbf{x})}\right)^{1/2} \int_{\mathbb{R}^3, (\boldsymbol{\xi} - \mathbf{w}) \cdot \mathbf{n} < 0} |(\boldsymbol{\xi} - \mathbf{w}(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x})| \, f(\mathbf{x}, \boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}$

ROUGHNESS OF SURFACES DUE TO ETCHING



FREE MOLECULE FLOW: LOW FREQUENCY LIMIT

If shuttlle velocity is small w.r.t. thermal velocity $|\tilde{\mathbf{w}}| = |\mathbf{w}|/\sqrt{2RT} \ll 1$.

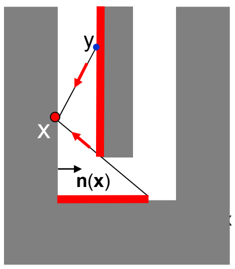
$\rho_w(\mathbf{x},t) \simeq \rho_{w0}(1+\rho_{w1}(\mathbf{x},t))$ and linearise with respect to ρ_{w1}:

assuming
$$w(\mathbf{x},t) = g(\mathbf{x})\dot{q}(t)$$
$$\rho_{w1}(\mathbf{x},t) = J(\mathbf{x})\dot{q}(t)$$

Limit case: $\frac{\omega}{\sqrt{2RT}} L \ll 1$

J(x): normalised flux of molecules at the wall

$$J(\mathbf{x}) = \sqrt{\pi}\tilde{\mathbf{g}}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) - \frac{1}{\pi}\int_{S+} J(\mathbf{y})\big(\mathbf{r}\cdot\mathbf{n}(\mathbf{x})\big)\big(\mathbf{r}\cdot\mathbf{n}(\mathbf{y})\big)\frac{1}{r^4}\mathrm{d}S + \frac{3}{2}\frac{1}{\sqrt{\pi}}\int_{S+}\big(\mathbf{r}\cdot\tilde{\mathbf{g}}(\mathbf{y})\big)\big(\mathbf{r}\cdot\mathbf{n}(\mathbf{x})\big)\big(\mathbf{r}\cdot\mathbf{n}(\mathbf{y})\big)\frac{1}{r^5}\mathrm{d}S$$



Quasi static approximation: radiosity equation

RADIOSITY COMPUTATION IN COMPUTER GRAPHICS

FORCES EXERTED ON THE SURFACE

Once J(x,ω) is available, compute tractions from similar BIE:

$$\mathcal{F}(\mathbf{p}) = t(\mathbf{x},\omega)\mathcal{F}(\dot{q})$$

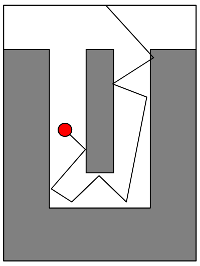
$$\begin{aligned}-\frac{\pi^{3/2}}{\rho_0 2\mathcal{R}T}\mathbf{t}(\mathbf{x},\omega) = {} & \left(1+\frac{1}{2}J(\mathbf{x},\omega)\right)\frac{\pi^{3/2}}{2}\mathbf{n} + \pi\tilde{g}_n(\mathbf{x})\mathbf{n}(\mathbf{x}) + \frac{\pi}{2}\tilde{\mathbf{g}}_t(\mathbf{x}) \\ & - \int_{S+} \mathbf{r}\Big(\mathbf{r}\cdot\mathbf{n}(\mathbf{x})\Big)\Big(\mathbf{r}\cdot\mathbf{n}(\mathbf{y})\Big)\frac{1}{r^5}J(\mathbf{y},\omega)T_4(i\tilde{\omega}r)\mathrm{d}S \\ & + 2\int_{S+} \mathbf{r}\Big(\mathbf{r}\cdot\tilde{\mathbf{g}}(\mathbf{y})\Big)\Big(\mathbf{r}\cdot\mathbf{n}(\mathbf{x})\Big)\Big(\mathbf{r}\cdot\mathbf{n}(\mathbf{y})\Big)\frac{1}{r^6}T_5(i\tilde{\omega}r)\mathrm{d}S\end{aligned}$$

$$\begin{aligned}-\frac{\pi^{3/2}}{\rho_0 2\mathcal{R}T}\mathbf{t}(\mathbf{x}) = {} & \left(1+\frac{1}{2}J(\mathbf{x})\right)\frac{\pi^{3/2}}{2}\mathbf{n}(\mathbf{x}) + \pi\tilde{g}_n(\mathbf{x})\mathbf{n}(\mathbf{x}) + \frac{\pi}{2}\tilde{\mathbf{g}}_t(\mathbf{x}) \\ & - \frac{3\sqrt{\pi}}{8}\int_{S+} \mathbf{r}\Big(\mathbf{r}\cdot\mathbf{n}(\mathbf{x})\Big)\Big(\mathbf{r}\cdot\mathbf{n}(\mathbf{y})\Big)\frac{1}{r^5}J(\mathbf{y})\mathrm{d}S \\ & + 2\int_{S+} \mathbf{r}\Big(\mathbf{r}\cdot\tilde{\mathbf{g}}(\mathbf{y})\Big)\Big(\mathbf{r}\cdot\mathbf{n}(\mathbf{x})\Big)\Big(\mathbf{r}\cdot\mathbf{n}(\mathbf{y})\Big)\frac{1}{r^6}\mathrm{d}S\end{aligned}$$

static limit case

•Frangi A., Ghisi A., Coronato L., On a deterministic approach for the evaluation of gas damping in inertial MEMS in the free-molecule regime,
•*Sensor & Actuators*, A 149, 21–28, 2009
•Frangi A., BEM technique for free-molecule flows in high frequency MEMS resonators, *Engineering Analysis with Boundary Elements,* 33, 493–498, 2009

TEST PARTICLE MONTE CARLO METHOD



$\Delta\mathbf{q}$ denotes the linear momentum change of one molecule due to one collision and $\mathbf{w}$ the instant velocity of the shuttle,

The total dissipation (energy transfer) induced by a single molecule before exiting the analysis domain through an in-flow surface is

$$d_i = \sum_c \Delta\mathbf{q}\cdot\mathbf{w}$$

for the *i-th in-flowsurface the number of incoming* molecules per unit time is

$$S_i \int_{\xi_n>0} \xi_n f_0 \,\mathrm{d}\xi = S_i \frac{\rho_0}{\sqrt{4\pi}}\sqrt{2\mathcal{R}T_0}$$

the average dissipation per unit-cycle *D due to the molecules entering* all the surfaces is

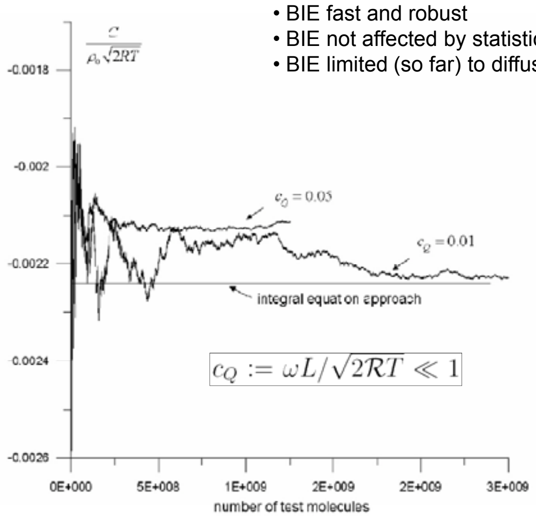
$$D = \frac{\rho_0\sqrt{2\mathcal{R}T_0}}{\sqrt{4\pi}}\frac{2\pi}{\omega}\sum_i (S_i\bar{d}_i)$$

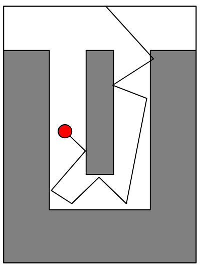
Finally the coefficient *B in the 1D model is*

$$B = \frac{\rho_0\sqrt{2\mathcal{R}T_0}}{\sqrt{4\pi}}\frac{2}{\omega_0^2 A^2}\sum_i (S_i\bar{d}_i)$$

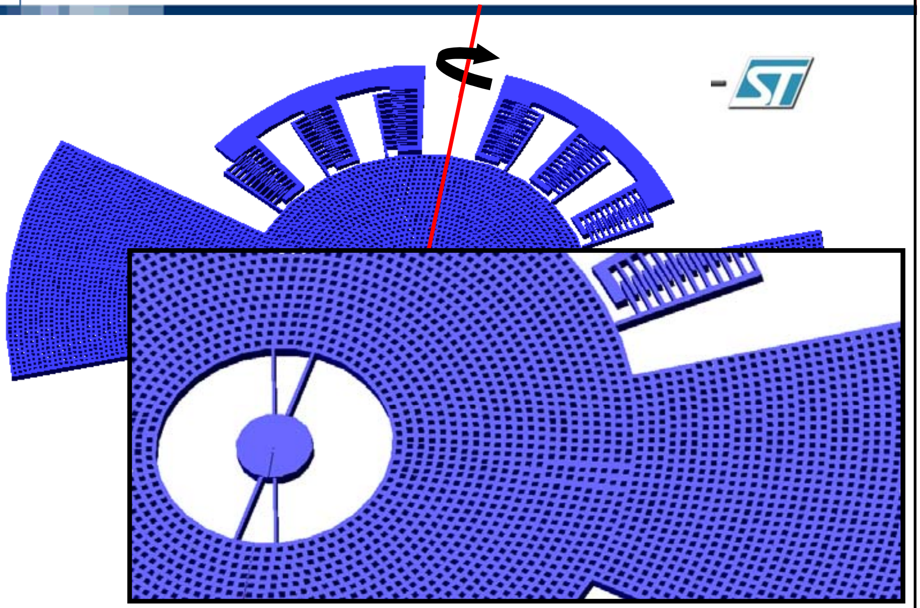
BIE vs TEST PARTICLE MONTE CARLO METHOD

- BIE independent of velocity profile imposed
- BIE fast and robust
- BIE not affected by statistical noise
- BIE limited (so far) to diffuse reflection boundary conditions

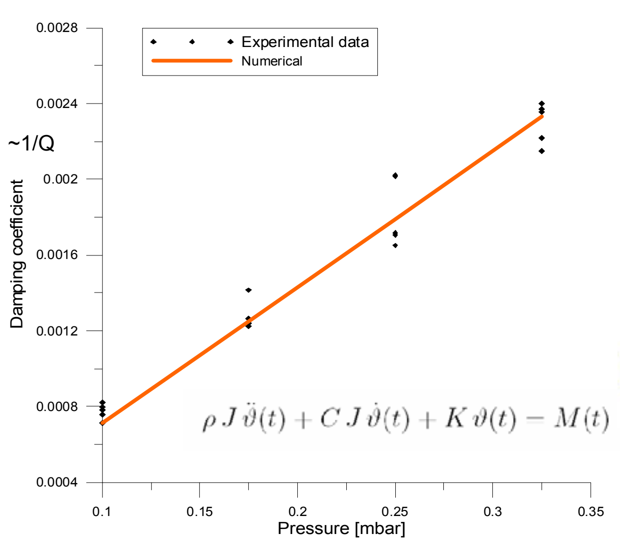




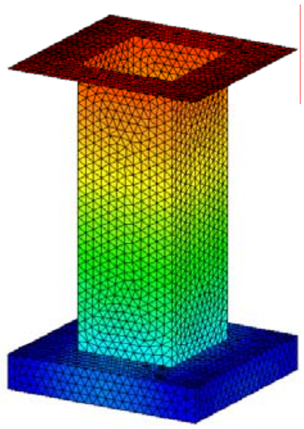
OUT-OF-PLANE ROTATIONAL RESONATOR



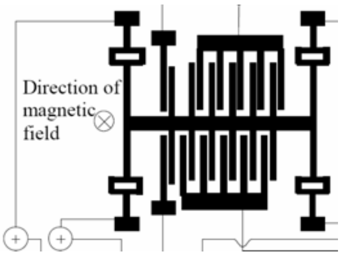
OUT-OF-PLANE ROTATIONAL RESONATOR

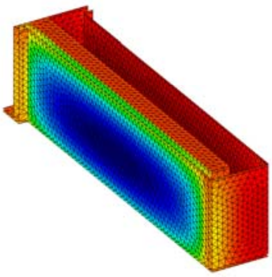


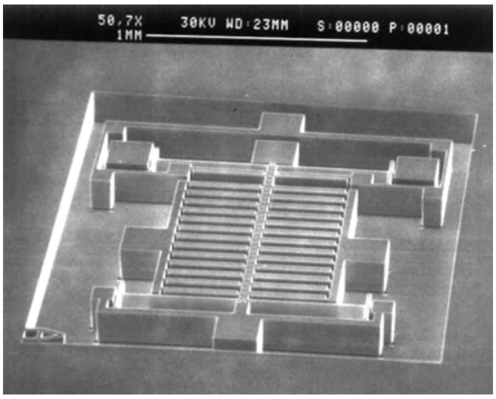
$$\rho J \ddot{\vartheta}(t) + C J \dot{\vartheta}(t) + K \vartheta(t) = M(t)$$



Qinetiq MAGNETOMETER

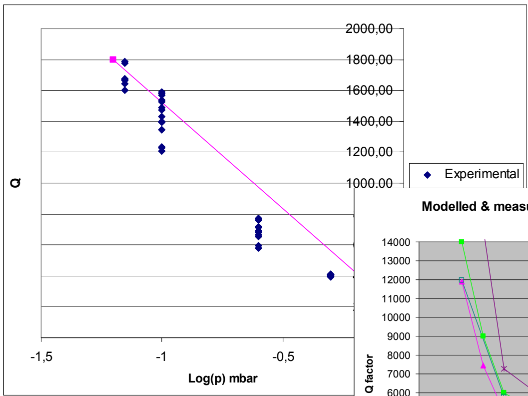


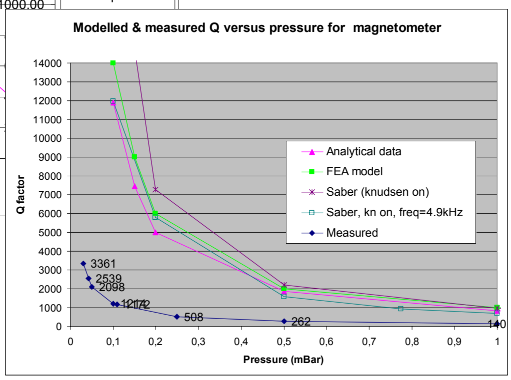




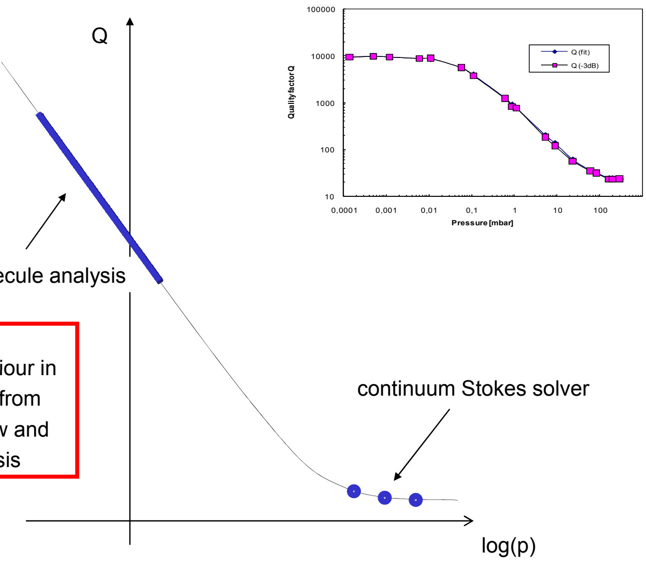
SEM of magnetometer structure

Qinetiq MAGNETOMETER





BRIDGING TECHNIQUES



Aim:
predict the behaviour in *transition regime* from free-molecule flow and slip Stokes analysis

FURTHER REFERENCES

http://www.stru.polimi.it/home/frangi/pubblicazioni.html

Thank you for your attention!