Fast BEM for Elastodynamic and Periodic Problems

N. Nishimura[1]

[1]Dept. Applied Analysis and Complex Dynamical Systems, Kyoto University, Japan

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Outline

BEM in elastodynamics

Boundary Element Methods in elastodynamics have a long history.

- Kupradze's work
  - Theoretical studies date back to 1930s(?)
  - Kupradze's functional equation method (collocate BIEs in the exterior of the domain under consideration)
- More recently
  - Early developments by Banaugh, Goldsmith (1963,4), Doyle(1966)
  - Rizzo, Cruse (1968)
  - Subsequent developments by many others.

BEM is suitable in elastodynamics because

- It can deal with exterior problems
  - No reflection from artificial boundaries
  - Hence no need for special techniques for no reflection
  - High accuracy
- Others

Applications

- NDE
- Earthquake engineering
- Others

But,

- Ordinary BEM cannot be applied to large problems because it scales as $O(N^2)$.

This problem was (almost) solved by fast BEMs such as FMBEM.

- FMM in Helmholtz was proposed by Rokhlin (1990) — diagonal forms
- Subsequent developments in EM community are impressive (Chew is the biggest contributor).
- FMM in elastodynamics
  - Chew's group (1997), Fukui(1998), Fujiwara(1998, 2000), Yoshida(2001),.....
  - To be discussed later.

Outline

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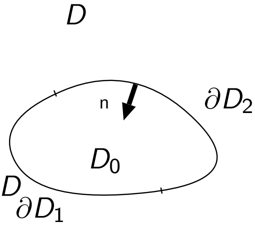
Formulation



Our problem is to solve

$$\mu\Delta u_i + (\lambda+\mu)u_{j,ij} + \rho\omega^2 u_i = 0 \quad \text{in } D$$

subject to

$$u_i = u_i^0 \quad \text{on } \partial D_1$$
$$Tu_i := C_{ijkl}u_{k,l}n_j = t_i^0 \quad \text{on } \partial D_2$$
$$\text{radiation condition for } u_i \quad \text{as } r = |x| \to \infty$$

u_i: displacement, (λ, μ): the Lamé constants,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

ρ: density, ω: frequency, u_i^0 and t_i^0: given data on ∂D_1 and ∂D_2.

The **solution** to this problem is well-known to have the following **integral representation**:

$$u_i(x) = \int_{\partial D} \Gamma_{ij}(x-y)t_j(y)dS_y - \int_{\partial D} \Gamma_{Iij}(x,y)u_j(y)dS_y, \quad x \in D$$

where Γ is the fundamental solution for elastodynamics given by:

$$\Gamma_{ij}(x) = \frac{1}{\mu}\left(G_T(r)\delta_{ij} + \frac{1}{k_T^2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left(G_T(r) - G_L(r)\right)\right)$$

$$G_{L,T} = \frac{e^{ik_{L,T}r}}{4\pi r}, \quad k_L = \sqrt{\frac{\rho}{\lambda+2\mu}}\omega, \quad k_T = \sqrt{\frac{\rho}{\mu}}\omega$$

Γ_{Iij} is the double layer kernel defined by

$$\Gamma_{Iij}(x,y) = \frac{\partial}{\partial y_l}\Gamma_{ik}(x-y)C_{klmj}n_m(y)$$

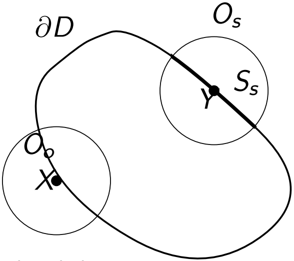
and $t_i = C_{ijkl}u_{k,l}n_j$ is the traction.

On the boundary:

$$\frac{u_i(x)}{2} = \int_{\partial D} \Gamma_{ij}(x-y)t_j(y)dS_y - \int_{\partial D} \Gamma_{Iij}(x,y)u_j(y)dS_y, \quad x \in \partial D$$

- This equation and the boundary conditions give the boundary integral equation for elastodynamics.
- Conventional approaches lead to $O(N^2)$ numerical methods.
- Fast methods such as FMM are needed in large scale problems.

FMMs for elastodynamics in frequency domain



We assume $x \in O_o$ and consider evaluating

$$V_i(x) = \int_{S_s} \Gamma_{ij}(x-y) t_j(y) dS_y - \int_{S_s} \Gamma_{Iij}(x,y) u_j(y) dS_y, \quad x \in \partial D$$

Key observation:

$$\Gamma_{ij}(x) = \frac{1}{\mu k_T^2} \left(e_{ipr} \frac{\partial}{\partial x_p} e_{jqr} \frac{\partial}{\partial y_q} G_T(r) + \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} G_L(r) \right) \qquad (1)$$

holds modulo Dirac's delta.

With this observation, together with the FMM tools for Helmholtz equation, one can formulate FMM for elastodynamics.

Use FMM tools for Helmholtz in $G_{T,L}(r)$:

$$\Gamma_{ij}(x) = \frac{1}{\mu k_T^2}\left(e_{ipr}\frac{\partial}{\partial x_p} e_{jqr}\frac{\partial}{\partial y_q} G_T(r) + \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} G_L(r) \right)$$

- Low frequency FMM

$$G(x-y) = \frac{ik}{4\pi}\sum_{n,m}\sum_{n',m'}(-1)^m I_n^{-m}(\overrightarrow{Yy})$$
$$(2n+1)(2n'+1)\hat{C}_{n,n'}^{m,m'}(\overrightarrow{YX}) I_{n'}^{m'}(\overrightarrow{Xx})$$
$$I_n^m(\overrightarrow{Ox}) = j_n(k|\overrightarrow{Ox}|) Y_n^m\left(\overrightarrow{Ox}/|\overrightarrow{Ox}|\right),$$

- Diagonal form

$$G(\mathbf{x}-\mathbf{y}) = \frac{ik}{(4\pi)^2}\int_{|\hat{\mathbf{k}}|=1} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{X})} T(\hat{\mathbf{k}},\overrightarrow{YX}) e^{-i\mathbf{k}\cdot(\mathbf{y}-\mathbf{Y})} dS,$$
$$T(\hat{\mathbf{k}},\overrightarrow{YX}) = \sum_{n=0}^{\infty} i^n(2n+1)P_n\left(\hat{\mathbf{k}}\cdot\overrightarrow{YX}/|\overrightarrow{YX}|\right) h_n^{(1)}\left(k|\overrightarrow{YX}|\right).$$

P_n: Legendre polynomial of the order n

FMMs in wave problems

- Low frequency FMM
  - series expansion of the kernel function
  - good accuracy for all frequencies
  - scales $O(N)$ in low frequency, but slows down in high frequency
- diagonal form
  - plane wave expansion of the kernel function
  - good accuracy for high frequency but breaks down in low frequency
  - scales $O(N \log^{(2)} N)$
- wideband FMM
  - use low freq FMM for smaller cells and diagonal forms for larger cells

Diagonal forms in elastodynamics

For example, **Diagonal forms** for elastodynamics can be obtained as follows:

$$V_i(x) = \int_{S_s} \Gamma_{ij}(x-y)t_j(y)dS_y - \int_{S_s} \Gamma_{Iij}(x,y)u_j(y)dS_y$$
$$= -\frac{1}{(4\pi)^2}\int_{|\hat{\mathbf{k}}|=1} \Big(k_L^2 \hat{k}_i e^{ik_L \hat{\mathbf{k}}\cdot(\mathbf{x}-\mathbf{X})} \tilde{L}^L(\hat{\mathbf{k}}, X)$$
$$+ k_T^2 e_{ipr}\hat{k}_p e^{ik_T \hat{\mathbf{k}}\cdot(\mathbf{x}-\mathbf{X})} \tilde{L}^T_r(\hat{\mathbf{k}}, X) \Big) dS_{\hat{\mathbf{k}}}$$

$\tilde{L}^L(\hat{\mathbf{k}}, X)$, $\tilde{L}^T_r(\hat{\mathbf{k}}, X)$: **coefficients of the local expansion**

$$\tilde{L}^L(\hat{\mathbf{k}}, X) = T(\hat{\mathbf{k}}, k_L, \overrightarrow{YX})\tilde{M}^L(\hat{\mathbf{k}}, Y)$$
$$\tilde{L}^T_r(\hat{\mathbf{k}}, X) = T(\hat{\mathbf{k}}, k_T, \overrightarrow{YX})\tilde{M}^T_r(\hat{\mathbf{k}}, Y) \quad \textbf{(M2L)}$$
$$T(\hat{\mathbf{k}}, k_{L,T}, \overrightarrow{YX}) = \sum_{n=0}^{\infty} i^n(2n+1)P_n\left(\hat{\mathbf{k}}\cdot\frac{\overrightarrow{YX}}{|\overrightarrow{YX}|}\right) h_n^{(1)}\left(k_{L,T}|\overrightarrow{YX}|\right)$$

$\tilde{M}^L(\hat{\mathbf{k}}, Y)$ and $\tilde{M}^T_r(\hat{\mathbf{k}}, Y)$ **multipole moments** for the diagonal forms.

Multipole moments for the diagonal forms are defined by

$$\tilde{M}^L(\hat{\mathbf{k}}, Y) = -ik_L \int_{S_s} e^{-ik_L \hat{\mathbf{k}}\cdot(\mathbf{y}-\mathbf{Y})} \left(\hat{k}_i t_i(y) + iu_i(y) k_L(\lambda n_i(y) + 2\mu \hat{k}_i \hat{k}_j n_j(y))\right) dS_y$$

$$\tilde{M}^T_r(\hat{\mathbf{k}}, Y) = -ik_T \int_{S_s} e^{-ik_T \hat{\mathbf{k}}\cdot(\mathbf{y}-\mathbf{Y})} \left(e_{ijr} \hat{k}_j t_i(y) + i\mu u_i(y) k_T \hat{k}_j (e_{ijr} \hat{k}_l + e_{ljr} \hat{k}_i) n_l(y)\right) dS_y$$

The shift formulae for $\tilde{M}$ and $\tilde{L}$ take the following forms:

$$\tilde{M}^L(\hat{\mathbf{k}}, Y_1) = \tilde{M}^L(\hat{\mathbf{k}}, Y_0)e^{-ik_L\hat{\mathbf{k}}\cdot\overrightarrow{Y_1Y_0}}$$

$$\tilde{M}_r^T(\hat{\mathbf{k}}, Y_1) = \tilde{M}_r^T(\hat{\mathbf{k}}, Y_0)e^{-ik_T\hat{\mathbf{k}}\cdot\overrightarrow{Y_1Y_0}} \quad \textbf{(M2M)}$$

$$\tilde{L}^L(\hat{\mathbf{k}}, X_1) = \tilde{L}^L(\hat{\mathbf{k}}, X_0)e^{ik_L\hat{\mathbf{k}}\cdot\overrightarrow{X_0X_1}}$$

$$\tilde{L}_r^T(\hat{\mathbf{k}}, X_1) = \tilde{L}_r^T(\hat{\mathbf{k}}, X_0)e^{ik_T\hat{\mathbf{k}}\cdot\overrightarrow{X_0X_1}} \quad \textbf{(L2L)}$$

Note that this formulation includes **4 moments** for every combination of (n, m).

- The scalar moment $\tilde{M}^L$ corresponds to **P wave**
- The vector moments $\tilde{M}^T$ represent **S wave**

in a natural manner.

Low frequency FMM and Wideband FMM in elastodynamics

- Low frequency FMM can be formulated similarly.

$$L^{L}_{n',m'}(X) = \sum_{n,m} (2n+1) \hat{C}^{m,m'}_{n,n'}(\overrightarrow{YX}, k_L) M^{L}_{n,m}(Y) \quad \text{(M2L), etc.}$$

$$\hat{C}^{m,m'}_{n,n'}(\overrightarrow{YX}, k_L) : \quad \text{coefficients}$$

- Wideband FMM needs conversions between the low frequency FMM and diagonal form

$$\tilde{M}^{L}(\hat{\mathbf{k}}, Y) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^{-n}(2n+1) Y^{m}_{n}(\hat{\mathbf{k}}) M^{L}_{n,m}(Y), \quad \text{etc.}$$

So far we have not tried a wideband FMM in elastodynamics. Examples of the use of such techniques are found in Maxwell's equations (to be shown later)

Numerical examples

Low frequency FMM

The first example is taken from Yoshida's thesis(2001) (9 years old).

Penny shaped crack subject to a plane incident P wave from below whose stress magnitude is p_0. Poisson's ratio: 0.25.

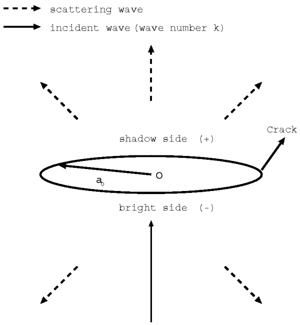


Figure: Scattering by a penny shaped crack

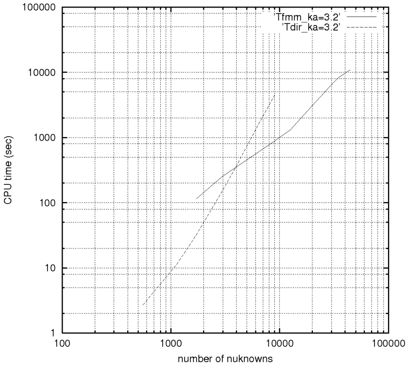


Figure: Total CPU time(sec) ($k_T a_0 = 3.2$)

Total CPU time (sec) vs the number of unknowns.
"Tdir_ka=3.2": CPU time for conventional BIEM,
"Tffm_ka=3.2": CPU time for FM-BIEM.
Machine: PC with DEC Alpha21264 (500MHz).

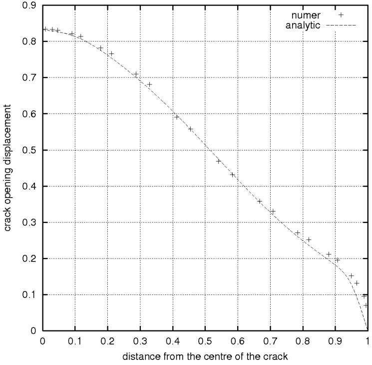


Figure: Numerical result and analytical solution ($k_T a_0 = 3.2$)

Magnitudes of the crack opening displacement obtained with FMM vs analytical solutions.

Diagonal forms

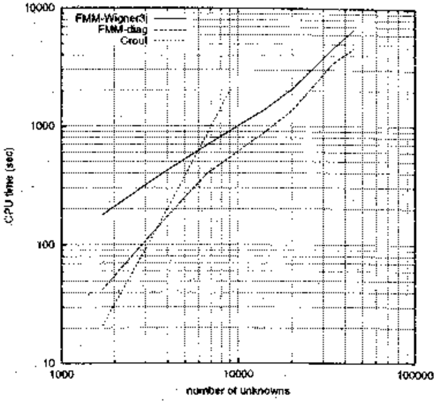


Fig. 1 CPU time (sec)

Figure: Total CPU time(sec) ($k_T a_0 = 5$)

- Taken from Yoshida et al.(2001)
- Single penny shaped crack problem subject to plane P wave. (wave number: $k_T a_0 = 5$)
- "crout": conventional BIEM, "FMM-Wigner3j": Low frequency FMM, "FMM-diag": diagonal form.
- Machine: PC with DEC Alpha21264 (600MHz).

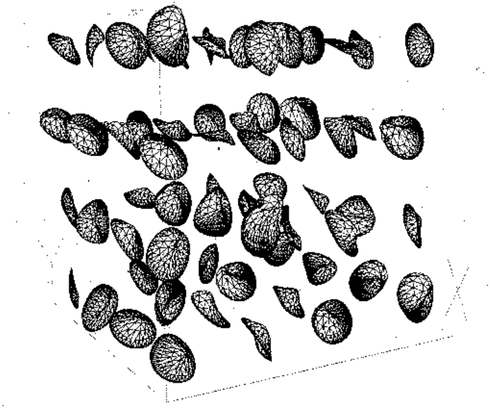


Fig.2 Crack opening displacements

Figure: Real part of the crack opening displacements ($k_T a_0 = 3$, 72,192DOF)

- Infinite elastic body which contains an **array of** $4 \times 4 \times 4$ **penny shaped cracks** of the same radius a_0
- Incident plane P wave from below
- The centres of cracks are located regularly at the interval of $4a_0$ in all the coordinate directions
- directions of the cracks are taken at random.

Bibliographical remarks

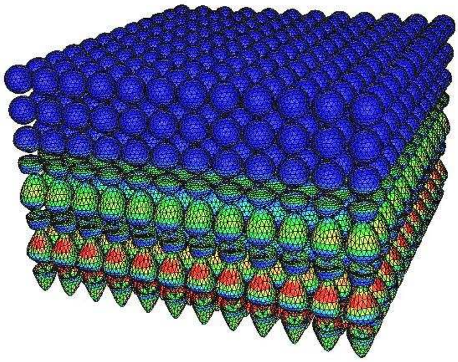
References on FMMs for elastodynamics are still scarce.

- 2D
  - **Chen et al.(1997)** Chew's group. Diagonal form. Many moments.
  - **Fukui (1998)** Low frequency FMM for elastodynamics using Galerkin's vector. This formulation uses 4 types of moments.
  - **Fujiwara (1998)** Low frequency FMM using 8 types of moments.
- 3D
  - **Fujiwara (2000)** Diagonal form approach in terms of 12 components of plane waves applied to low frequency problems related to earthquake.
  - **Yoshida et al.(2001a)** (also available in English (Yoshida's thesis(2001)) Low frequency FMM for crack problems in 3D. 4 moments.
  - **Yoshida et al.(2001b)** Diagonal form version. Immature error control.
  - **Chaillat et al.(2007,8)** Diagonal forms
  - **Sanz et al.(2008)** SPAI preconditioner
  - **Tong and Chew (2009)** Use of Nyström's method
  - **Chaillat et al.(2009)** Multi-domain problems and applications to seismological problems.

Outline

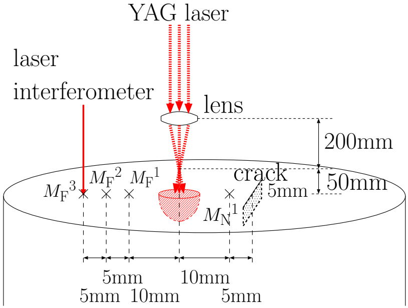
Time domain

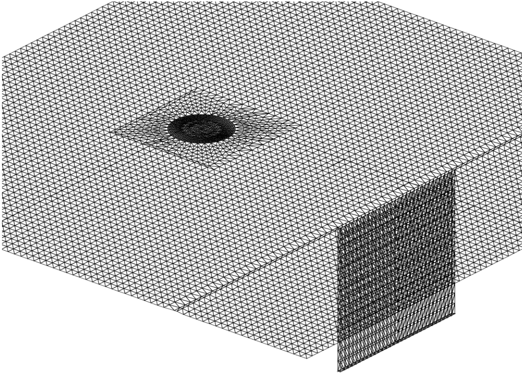
- Time domain FMM in elastodynamics is a possibility.
- We here present a few numerical examples.



- 1152 spherical holes (Otani and Nishimura (2004))
- 1,105,290 spatial DOF, 200 time steps
- CPU time: 10H 47Min

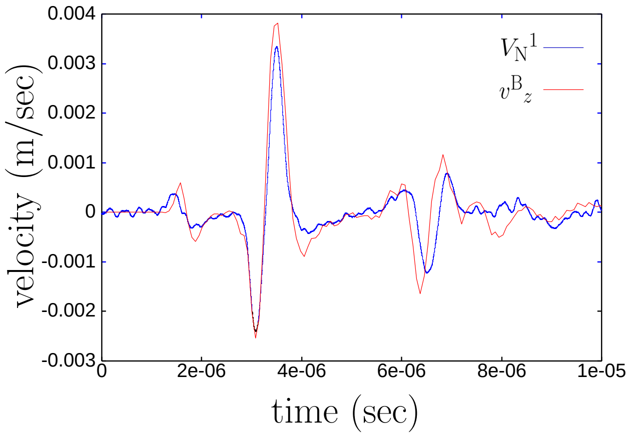
NDT application: laser ultrasonics





- Specimen (Aluminium aloy)
- rectangular surface crack length=10mm, depth=5mm, distance from the laser spot=15mm
- Energy of the pulse laser: 19mJ

- 48900DOF
- 134 time steps
- solved with Fast BIEM
- Yoshikawa et al.(2007)



$V_{\mathrm{N}}{}^{1}$: Measured velocity at $M_{\mathrm{N}}{}^{1}$

$v^{\mathrm{B}}{}_{z}$: Computed velocity at $M_{\mathrm{N}}{}^{1}$ obtained with Fast BIEM

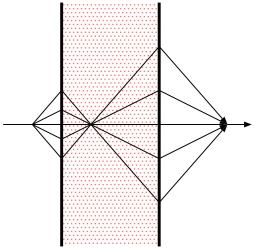
Outline

Periodic FMM for Wave Problems

- There are a number of interesting applications which are reduced to periodic boundary value problems for wave problems.

Examples — Maxwell's equations

- Metamaterials
  - Periodic structure with metalic inclusions
  - The periodicity is much smaller than the wavelength.
  - One may control the optical properties of the composite quite freely.
  - Negative refractive index, Super-lens, Cloaking



super-lens

Examples (cont.)

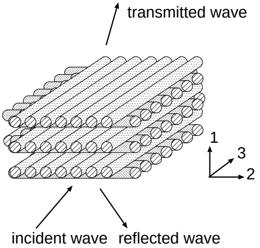
- Photonic crystals
  - Periodic structure of dielectric materials
  - The periodicity is of the order of the wavelength of light
  - Works as a waveguide.
  - Interesting properties such as frequency selectivity, localised modes, etc.

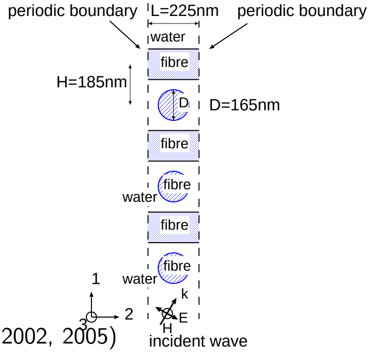
  item Optical devices
  - digital camera

We have developped a 2-periodic FMM for Maxwell's equations in 3D.
(Periodic in 2 directions. Scattering in the other direction.)
J. Comp. Phys. (2008), Waves in Random and Complex Media (2009)

Numerical example — skin of worm

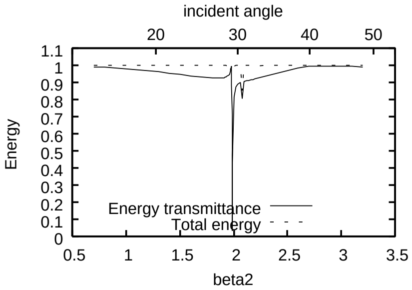
Maxwell's equation in 3D

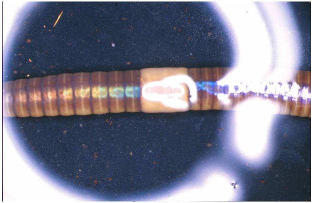




- Modelled by Miyamoto and Kosaku (2002, 2005)
- 6 layers of glass fibres in water
- Diameter of the fibre: 165nm
  distance between fibre centres in the x_1 direction: 185nm,
  Period length: $L_2 = L_3 = 225$mn
- Incident plane wave
  - Incident wave length: 475nm
  - Incident angle from x_2 axis: varies around 30°.

Skin of worm — energy transmittance





http://nkiso.u-tokai.ac.jp/form/event/ssf/sympo54/struct_color/struct_color.htm

- Wideband FMM, 107,568 DOF
- Agrees with experiments.

It is of interest to investigate elastic counterparts of photonic crystals and metamaterials.

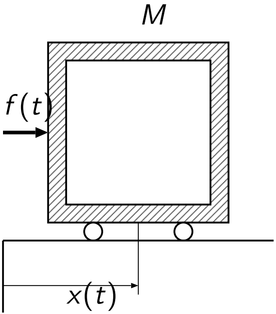
- Phononic crystals (775 papers found in Scopus)
  - Bandgap structures
  - Can we guide elastic waves as freely as we wish?
- Acoustic metamaterials (64 papers found in Scopus)
  - Negative density
  - Negative modulus of elasticity
  - Negative refractive index

principle of metamaterials

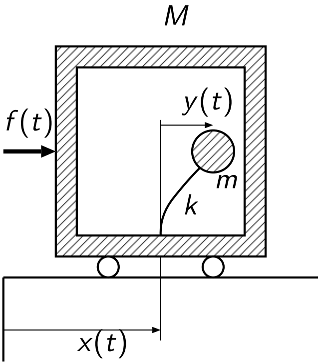
How can you measure mass? Shake it!

$$M\ddot{x} = f \qquad (x = Xe^{-i\omega t}, \quad f = Fe^{-i\omega t})$$

$$M = -\frac{F}{\omega^2 X}$$



If you have internal structures

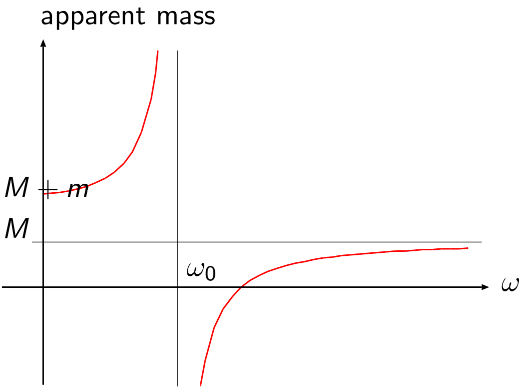


$$M\ddot{x} = f + ky$$
$$m(\ddot{x} + \ddot{y}) = -ky$$
$$x = Xe^{-i\omega t}, \quad y = Ye^{-i\omega t},$$
$$f = Fe^{-i\omega t}$$

One will conclude $M = -\frac{F}{\omega^2 X}$ if one does not know the inside of the box.

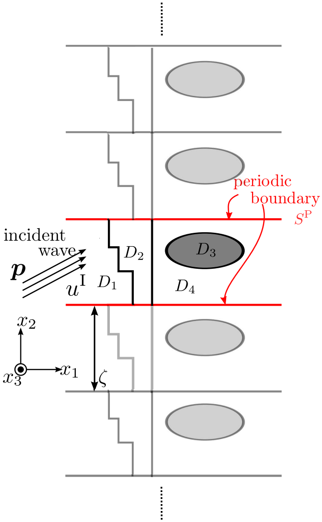
$$-\frac{F}{\omega^2 X} = M + \frac{m}{1 - \left(\frac{\omega}{\omega_0}\right)^2}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

negative mass



- One may utilise the resonance to control the apparent material properties in metamaterials with internal structure.
- negative mass and negative elastic modulus may lead to propagating waves with $c = \sqrt{\mu/\rho}$.

Periodic FMM in Elastodynamics



governing eq.

$$\mu^m u_{i,jj} + (\lambda^m + \mu^m) u_{j,ij} + \rho^m \omega^2 u_i = 0 \quad x \in D_m$$

B.C.

displacement u_i and traction t_i

are continuous across $\partial D_k \cap \partial D_m$

Radiation condition for $u_i - u_i^{\mathrm{I}}$ as $r = |\mathbf{x}| \to \infty$

periodic B.C.

$$u_i(x_1, \frac{\zeta}{2}, x_3) = e^{\mathrm{i}\beta_2} u_i(x_1, -\frac{\zeta}{2}, x_3)$$

$$\frac{\partial u_i}{\partial x_2}(x_1, \frac{\zeta}{2}, x_3) = e^{\mathrm{i}\beta_2} \frac{\partial u_i}{\partial x_2}(x_1, -\frac{\zeta}{2}, x_3)$$

$$u_i(x_1, x_2, \frac{\zeta}{2}) = e^{\mathrm{i}\beta_3} u_i(x_1, x_2, -\frac{\zeta}{2})$$

$$\frac{\partial u_i}{\partial x_3}(x_1, x_2, \frac{\zeta}{2}) = e^{\mathrm{i}\beta_3} \frac{\partial u_i}{\partial x_3}(x_1, x_2, -\frac{\zeta}{2})$$

where $\beta_i = k_{\mathrm{L,T}} p_i \zeta, \quad k_{\mathrm{L,T}}\zeta \neq 2n\pi \pm \beta_i$

BIE for periodic B.V.P

Boundary Integral Equation for periodic B.V.P (ordinary BIE for inclusions)

$$\frac{1}{2}\Big[u_i(x)+\alpha t_i(x)\Big] = \Big[u_i^{\mathrm{I}}+\alpha t_i^{\mathrm{I}}\Big]$$
$$+\,\mathrm{p.v.}\int_{\partial D}\Big[\Gamma_{ij}^{\mathrm{P}}(\mathbf{x}-\mathbf{y})+\alpha T_{ik}\Gamma_{kj}^{\mathrm{P}}(\mathbf{x}-\mathbf{y})\Big]t_j(y)dS_y$$
$$-\,\mathrm{p.f.}\int_{\partial D}\Big[\Gamma_{Iij}^{\mathrm{P}}(\mathbf{x}-\mathbf{y})+\alpha T_{ik}\Gamma_{Ikj}^{\mathrm{P}}(\mathbf{x}-\mathbf{y})\Big]u_j(y)dS_y \quad \text{Burton-Miller}$$

periodic Green's function

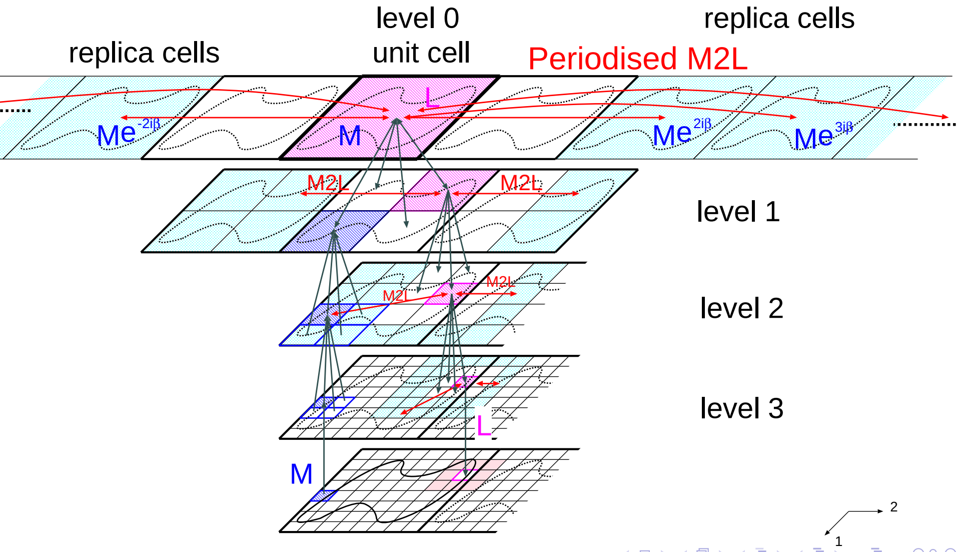
$$\Gamma_{ij}^{\mathrm{P}}(\mathbf{x}-\mathbf{y}) = \lim_{R\to\infty}\sum_{\boldsymbol{\omega}\in\mathcal{L}(R)}\Gamma_{ij}(\mathbf{x}-\mathbf{y}-\boldsymbol{\omega})e^{\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\omega}}$$

$$\mathcal{L}(R)=\{(0,\omega_2,\omega_3)|\omega_2=p\zeta,\omega_3=q\zeta,\ |p|,|q|\le R,\ p,q\in\mathbb{Z}\}$$

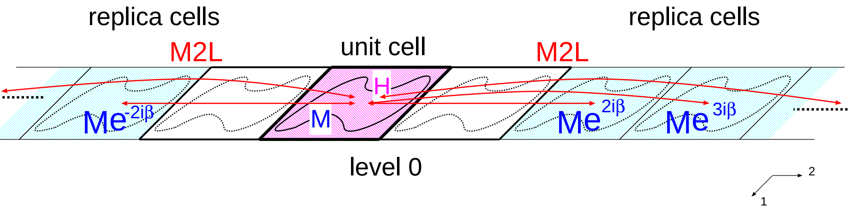
fundamental solution for 3D elastodynamics

$$\Gamma_{ij}(\mathbf{x}-\mathbf{y})=\frac{1}{\mu}\left[\frac{e^{\mathrm{i}k_{\mathrm{T}}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\delta_{ij}+\frac{1}{k_{\mathrm{T}}^2}\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{e^{\mathrm{i}k_{\mathrm{T}}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}-\frac{e^{\mathrm{i}k_{\mathrm{L}}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\right)\right]$$

Algorithm for periodic FMM



M2L in Periodic FMM (low freq. FMM)



mutipole moments of the replica cell(ω)

$$M_{n,m}(\omega) = M_{n,m}(O)e^{\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\omega}}$$

M2L formula

$$L_{n',m'}(O) = \sum_n \sum_m (2n+1)\hat{C}^{m,m'}_{n,n'}(-\boldsymbol{\omega})M_{n,m}(\hat{\mathbf{k}}, O)e^{\mathrm{i}\boldsymbol{\beta}\cdot\omega}$$

Periodisation

$$L_{n',m'}(O) = \sum_n \sum_m (2n+1)\sum_{\boldsymbol{\omega}\in\mathcal{L}'}\left(\hat{C}^{m,m'}_{n,n'}(-\boldsymbol{\omega})e^{\mathrm{i}\boldsymbol{\beta}\cdot\omega}\right) M_{n,m}(O)$$

periodised M2L

We compute the influence from far replica cells by the ‘periodised M2L’ given by:

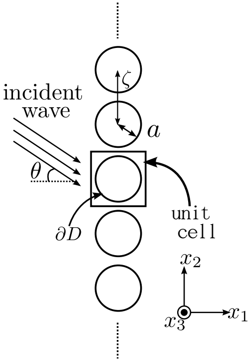
$$L^{\mathrm{L}}_{n,m}(O) = \sum_{n}^{\infty} \sum_{m=-n}^{n} (2n+1) \left(\sum_{\omega \in \mathcal{L}'} \hat{C}^{m',m}_{n',n}(-\boldsymbol{\omega}, k_{\mathrm{L}}) e^{\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\omega}} \right) M^{\mathrm{L}}_{n',m'}(O)$$

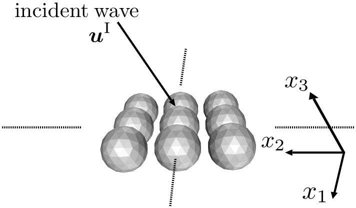
$$= \sum_{n}^{\infty} \sum_{m=-n}^{n} (2n+1) \hat{C}^{\mathrm{P}\,m',m}_{n',n}(k_{\mathrm{L}}) M^{\mathrm{L}}_{n',m'}(O)$$

$$L^{\mathrm{T}}_{r;n,m}(O) = \sum_{n}^{\infty} \sum_{m=-n}^{n} (2n+1) \hat{C}^{\mathrm{P}\,m',m}_{n',n}(k_{\mathrm{T}}) M^{\mathrm{T}}_{r;n',m'}(O)$$

- Evaluation of $\hat{C}^{\mathrm{P}\,m',m}_{n',n}$ is reduced to the computation of the lattice sum $\sum_{\omega \in \mathcal{L}'} h^{(1)}_n(|\boldsymbol{\omega}| k_{\mathrm{L,T}}) Y^m_n \left(-\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \right) e^{\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\omega}}$, which is extremely slow to converge.
- We use Fourier analysis to compute the lattice sums.

numerical examples

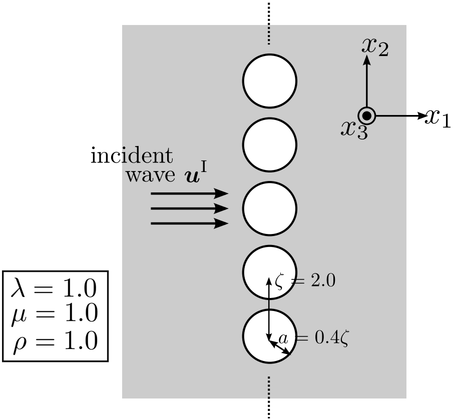




- scattering of plane waves by a doubly periodic layer of elastic spheres (Isakari and Nishimura (2010))
- collocation method, piecewise constant element (18000 elements, 108,000DOF)
- Flexible GMRES (criterion of convergence: 10^{-5})
- preconditioner: part of matrix computed directly in the FMM algorithm

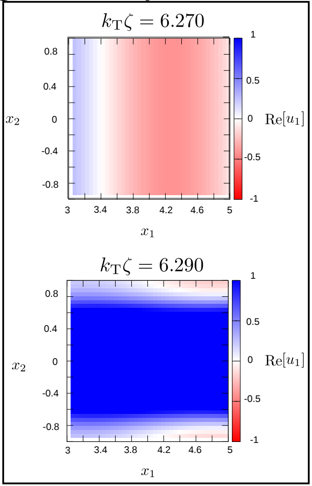
numerical examples: scattering by holes

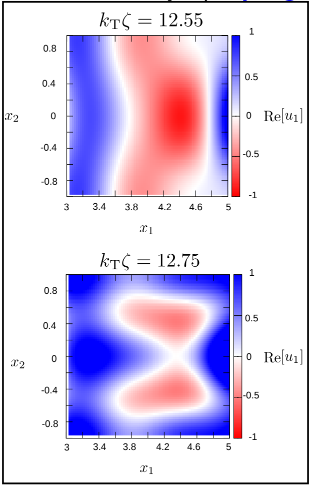
- Incident angle is $\theta = 0.0^\circ$, P-wave incidence.



numerical examples: scattering by holes

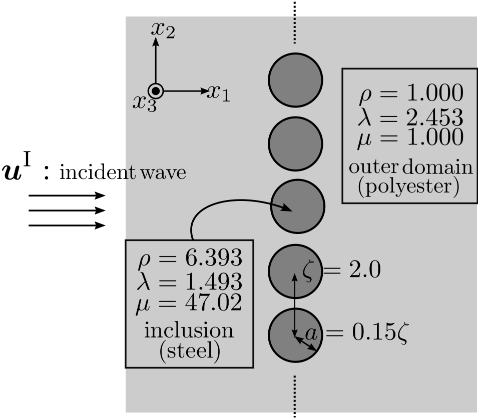
The far field patterns in the transmission side for $k_T\zeta$s slightly smaller and larger than $k_T\zeta = 2\pi$ and 4π differ considerably. (Rayleigh's anomaly)



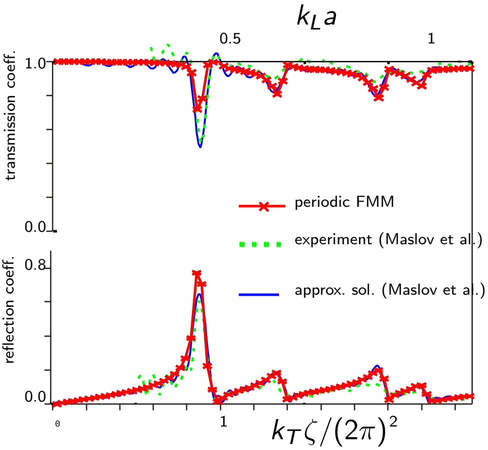


numerical examples: scattering by elastic inclusions

- We have computed reflection and transmission coefficients in the frequency range of $\omega = 0.0 \sim 8.2$
- Incident angle is $\theta = 0.0^\circ$, P-wave incidence.



numerical examples: scattering by elastic inclusions



transmission and reflection coefficient

- Agrees with results in reference.
- Anomalies of **resonance type** are seen.

Outline

Conclusion

- Use of FMM in elastodynamics is effective in frequency domain.
- Time domain FMM in elastodynamics is a possibility.
- Periodic FMM is extended to elastodynamics.
- FMM in elastodynamics deserves more attention!