INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 37, 107–121 (1994)

BOUNDARY INTEGRAL EQUATIONS FOR THIN BODIES

GUNA KRISHNASAMY

PDA Engineering, 2975 Redhill Avenue, Costa Mesa, CA 92626, U.S.A.

FRANK J. RIZZO AND YIJUN LIU

Center for Nondestructive Evaluation and Department of Aerospace Engineering and Engineering Mechanics, Iowa State University, Ames, IA 50011, U.S.A.

SUMMARY

A boundary integral equation formulation for thin bodies which uses CBIE (conventional BIE) only is well known to be degenerate. A mixed formulation for a thin rigid scatterer which combines CBIE and HBIE (hypersingular BIE) is motivated by examining the discretized form of the integral equations, and this formulation is shown to be non-degenerate for thin non-rigid inclusion problems. A near-singular integration procedure, useful for singular integrals as well, is presented. Finally, numerical examples for acoustic wave scattering from rigid and soft scatterers are presented.

INTRODUCTION

Boundary integral equations (BIEs) and the boundary element method (BEM) for their solution have been either avoided, or employed with considerable difficulty, whenever the bodies at issue are very slender or thin, i.e. when the volume enclosed by the bounding surface is small. For such bodies, if the BIE/BEM has been used at all, it is in the form of a shell or plate type theory. The well-understood limitations or approximations in such theories are often unacceptable. Problems of this type are sometimes simply conceded to the realm of more effective treatment by finite elements, regardless of how effective that treatment may be. Similar remarks pertain for bulky or thick bodies containing thin inclusions or thin voids.

For the case of a very thin void in a solid, modelling the void as a crack is often useful [1]. Crack models generally are easier to handle by any method, numerical or analytical, than very thin cracklike voids, but this paper is concerned with situations where the crack model is not very appropriate. Examples are the fields associated with notches, rough cracks, tapered shapes such as aerofoils and thin inclusions.

The difficulties with the basic boundary integral approach for thin bodies are twofold: (1) the BIEs often become nearly degenerate when parts of the surface are separated only by a small distance or thickness, and (2) the task of achieving even ordinary arithmetic accuracy with the BEM, in the face of the near-singular integrals which arise in such problems, is very difficult.

It should be noted, before saying more, that success with difficulty (2) in light of (1) allows success with thin shapes only up to a point. The near-degeneracy of conventional BIE's usually leads to ill-conditioning of the algebraic equations even with arbitrarily-accurate numerical quadrature, when shapes become very thin. In the limit of zero-thickness such as a crack-model, the BIE degenerates, as is well known.

0029-5981/94/010107-15$12.50

Received 15 September 1992
Revised 28 January 1993

It is useful for the present discussion to consider thin-body problems in two classifications: (a) the thin body is in contact with a medium exterior to it which is of infinite extent or at least is itself bulky or not thin, and (b) the thin body exists by itself, without contact with an exterior medium, and loads (and boundary conditions) are applied directly to the thin body.

In this paper we present a boundary integral formulation for time harmonic wave scattering from thin bodies which is free of both difficulties (1) and (2) for problems in classification (a). Applications include scattering of acoustic or elastic wave scattering from thin voids, rough or partially closed cracks thin inclusions, solids joined by a weld or glue, acoustic scattering or radiation into a fluid from thin elastic structures (e.g. appendages from submarines). The formulation involves using a conventional BIE on 'one side of the thin dimension' and a hypersingular BIE (HBIE) on the 'other side' [2–4]. Each BIE refers to the exterior medium. This combination of two types of integral equations takes care of difficulty (1) for problems in classification (a). (Static versions of scattering problems are defined in the zero-frequency limit). CBIEs or HBIEs may be used alone or in combination for the interior (thin) medium.

Next we present a numerical integration procedure for accurate evaluation of near-singular integrals. This procedure takes care of difficulty (2) and gives accuracy comparable to that presently possible for truly singular integrals or for regular integrals. It is not restricted to flat surfaces, and it works for strongly singular or hypersingular integrals as a special case. Also, it works for problems in both classification (a) and (b).

Specific ideas in this paper are developed for problems of scattering of time-harmonic waves from thin scatterers surrounded by a fluid where the mathematical issues are similar to (the scalar analogue of those of) scattering of elastic waves from thin voids. Conditioning and accuracy issues involving thinness are the same if the scatterers are voids in the solid or rigid scatterers in the fluid, or they may be thin inclusions of different wave-propagating material in either case. Numerical examples are given for scattering of acoustic waves from various thin pancake-shaped rigid scatterers and from a ripple-shaped scatterer which simulates a thin but rough scatterer in the fluid. There is also an example of scattering from thin fluid inclusions with different densities and wave speeds from the host fluid.

For problems in classification (b), the CBIE/HBIE-combination strategy, as used above for the medium exterior to the thin body, is not effective in preventing near-degeneracy if applied to the thin body itself. If known loads (displacements) are applied to each surface on either side of a thin plate, for example, a traction (displacement) boundary value problem, there will be near degeneracy with any of the BIE formulas alone or in combination if collocation is attempted on both surfaces. This is not surprising since there is no clearly meaningful problem in the limit as the thin body vanishes. However, it is interesting to note that the mixed problem, i.e. traction prescribed on one side and the displacement on the other, is non-degenerate with any of the BIE formulas. This point will be discussed further subsequently.

What then is to be done with thin bodies by themselves loaded with known tractions as is commonly done in the most important applications? It is possible to solve such problems with BIEs if they are written and collocation is done on only one surface, e.g. the midsurface of a thin plate, and if another relation involving, say, the unknown displacements on each surface can be written. Such a relation might come from apparent symmetries in a given problem, or come from a reasonable assumption based on the thinness of the plate. Such an assumption may at first glance be equivalent to a plate theory, but it may in fact be less restrictive, involving as it does only surface quantities, and not requiring a statement about variation of (or neglect of) certain field quantities throughout the thickness. Research into this matter for flat plate-like structures has begun and will be reported elsewhere. In this paper, however, we confine attention to the more straightforward thin-body problems in classification (a).

Finally, the formulation presented here is independent of scatterer thickness and can be used for bulky scatterers, thin scatterers, as well as cracks. Also, discretization needed to model the scatterer does not depend (directly) on scatterer thickness. Most importantly, the formulation used here is exact in the sense no plate or shell theories are used.

ACOUSTIC SCATTERING FROM THIN BODIES

CBIE formulation

A CBIE for scattering of time harmonic acoustic waves exterior to a scatterer of arbitrary shape can be written [5, 6]

$$\frac{1}{2}\phi(\xi_0) + \int\!\!\!\!\!\!-_S \frac{\partial G(x,\xi_0)}{\partial n(x)}\phi(x)\,\mathrm{d}S(x) = \int_S G(x,\xi_0)q(x)\mathrm{d}S(x) + \phi^{\mathrm{I}}(\xi_0) \tag{1}$$

where the total field $\phi = \phi^{\mathrm{I}} + \phi^{\mathrm{S}}$ is the sum of the (known) incident and scattered field, x and ξ_0 are points on the scatterer surface S, $n(x)$ is the normal at x on S, $q(x) = \partial\phi(x)/\partial n(x)$, $G(x, \xi_0) = \exp(\mathrm{i}kr)/4\pi r$, $r = |x - \xi_0|$, k is the wave number and $i = \sqrt{-1}$. In this section we are concerned only with the region exterior to the scatterer, and for definiteness here regard $q(x)$ as a known quantity (e.g. zero for a rigid scatterer). Equation (1) is to be solved for $\phi(x)$ from which other desired quantities may be readily determined. The scattered field is assumed to satisfy the radiation condition at infinity and

$$\int\!\!\!\!\!\!-_S \frac{\partial G(x,\xi_0)}{\partial n(x)}\phi(x)\mathrm{d}S(x) = \lim_{S_\varepsilon\to 0}\int_{(S-S_\varepsilon)} \frac{\partial G(x,\xi_0)}{\partial n(x)}\phi(x)\mathrm{d}S(x) \tag{2}$$

where S_ε is a small disk of constant radius epsilon centred at ξ_0.

To address the degeneracy of the integral equations for thin shapes (difficulty 1), assume for now that all integrals can be computed exactly. Consider a thin scatterer as shown in Figure 1, where S^+ and S^- are the two surfaces of the scatterer. The points and boundary variables on these two surfaces are identified by their superscripts. When collocating at a point ξ_0^+ (ξ_0^-) on S^+ (S^-) the CBIE, equation (1), can be written as

$$\frac{1}{2}\phi^{\pm}(\xi_0^{\pm}) + \int_{S^+} \frac{\partial G(x,\xi_0^{\pm})}{\partial n(x^+)}\phi^+(x)\mathrm{d}S(x) + \int_{S^-} \frac{\partial G(x,\xi_0^{\pm})}{\partial n(x^-)}\phi^-(x)\mathrm{d}S(x)$$
$$= \int_{S^+} G(x,\xi_0^{\pm})q^+(x)\mathrm{d}S(x) + \int_{S^-} G(x,\xi_0^{\pm})q^-(x)\mathrm{d}S(x) + \phi^{\mathrm{I}\pm}(\xi_0^{\pm}) \tag{3}$$

Here the first (second) integral is to be interpreted as in equation (2) when the collocation point is on S^+ (S^-). In the limiting case as S^- goes to S^+, i.e. when S^- and S^+ occupy the same position,

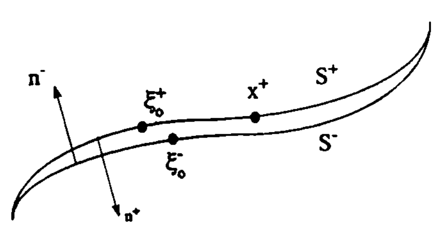

Figure 1. An arbitrarily shaped thin scatterer

equation (3) becomes

$$\frac{1}{2}\Sigma\phi(\xi_0) + \int_{S^+} \frac{\partial G(x,\xi_0)}{\partial n(x)}\Delta\phi(x)\,dS(x) = \int_{S^+} G(x,\xi_0)\Sigma q(x) + \phi^{I+}(\xi_0) \quad (4)$$

where $\phi^+(\xi_0^+) + \phi^-(\xi_0^+) = \Sigma\phi(\xi_0)$, $\phi^+(\xi_0^+) - \phi^-(\xi_0^+) = \Delta\phi(\xi_0)$ and $q^+(x) + q^-(x) = \Sigma q(x)$.

Equation (3) with collocation point at ξ_0^+ is *nearly* the same as that got with collocation point at ξ_0^- for small separation of S^+ and S^-. Both of these are the same, equation (4), in the limit as S^- goes to S^+. Clearly, attempts to compute with equation (3) for such cases are bound to suffer from the near-degeneracy and ultimately fail for small enough thickness.

It is important to note that equation (4) with variables $\Sigma\phi$, $\Delta\phi$ and Σq are defined only for the zero-thickness case. However, equation (3) with collocation point on S^- or S^+, have valid interpretations for bulky scatterers and very thin scatterers, even though, as we have observed, these equations are nearly degenerate for the latter case.

Additional insight into this near degeneracy can be gained by examining, *in discretized form*, equations (3) for a uniformly thin rigid body ($q^+ = q^- = 0$) whose two surfaces are assumed flat for convenience, i.e.

$$\begin{bmatrix} \frac{1}{2}I & \sim\frac{1}{2}I \\ \sim\frac{1}{2}I & \frac{1}{2}I \end{bmatrix}\begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix} = \begin{bmatrix} \Phi^{I+} \\ \Phi^{I-} \end{bmatrix} \quad (5)$$

and where I is the identity matrix and the columns represent discrete nodal values of Φ and Φ^I on S^+ and S^-. The first (second) row of the above matrix equation corresponds to equation (3) with collocation point on S^+ (S^-). Here elements on S^- are assumed to be the mirror image of those on S^+ and the terms $\sim(1/2)I$ are nearly equal to $(1/2)I$ for small thicknesses. In the limit as S^- goes to S^+, equation (4) becomes

$$\begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & \frac{1}{2}I \end{bmatrix}\begin{bmatrix} \Sigma\Phi \\ \Sigma\Phi \end{bmatrix} = \begin{bmatrix} \Phi^{I+} \\ \Phi^{I-} \end{bmatrix} \quad (6)$$

corresponding to the discretized form of equation (3). Note $\Delta\phi$ does not appear since $\partial G(x,\xi_0)/\partial n$ is zero for flat S^+.

HBIE formulation

The acoustic scattering problem above when formulated as a HBIE [7–12], takes the form

$$\frac{1}{2}q(\xi_0) + \int_S \frac{\partial^2 G(x,\xi_0)}{\partial n(x)\partial n(\xi_0)}\phi(x)\,dS(x) = \int_S \frac{\partial G(x,\xi_0)}{\partial n(\xi_0)}q(x)\,dS(x) + q^I(\xi_0) \quad (7)$$

where S, n, x, ξ_0 are as defined before. The two dashes through the integral sign implies that the integral is interpreted as the Hadamard finite part [13]. The above formulation has been used for non-slender scatterers [14, 15].

For a thin scatterer, Figure 1, the HBIE when collocating on S^+ (S^-) is

$$\frac{1}{2}q^{\pm}(\xi_0^{\pm}) + \int_{S^+} \frac{\partial^2 G(x,\xi_0^{\pm})}{\partial n(x^+)\partial n(\xi_0^{\pm})}\phi^+(x)\,dS(x) + \int_{S^-} \frac{\partial^2 G(x,\xi_0^{\pm})}{\partial n(x^-)\partial n(\xi_0^{\pm})}\phi^-(x)\,dS(x)$$

$$= \int_{S^+} \frac{\partial G(x,\xi_0^{\pm})}{\partial n(\xi_0^{\pm})}q^+(x)\,dS(x) + \int_{S^-} \frac{\partial G(x,\xi_0^{\pm})}{\partial n(\xi_0^{\pm})}q^-(x)\,dS(x) + q^{I\pm}(\xi_0^{\pm}) \quad (8)$$

Here the first (second) integral is interpreted as a finite part when the collocation point is on S^+

(S^-) and the third (fourth) integral is as defined in equation (2). For a zero-thickness scatterer where S^+ and S^- are the same (but with different normal directions) equation (8) can be written as an integral equation over S^+:

$$\frac{1}{2}\Delta q(\xi_0) + \int_{S^+} \frac{\partial^2 G(x,\xi_0)}{\partial n(x)\,\partial n(\xi_0)}\Delta\phi(x)\,\mathrm{d}S(x) = \int_{S^+} \frac{\partial G(x,\xi_0)}{\partial n(\xi_0)}\Sigma q(x)\,\mathrm{d}S(x) + q^{\mathrm{I}}(\xi_0) \qquad (9)$$

The discretized form of equations (8) with collocation point on S^+ and S^- for a pancake-shaped scatterer as described earlier is

$$\begin{bmatrix} C & \sim -C \\ \sim -C & C \end{bmatrix} \begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix} = \begin{bmatrix} Q^{\mathrm{I}+} \\ Q^{\mathrm{I}-} \end{bmatrix} \qquad (10)$$

where C is a matrix involving integrals of known quantities and $Q^{\mathrm{I}+}$ and $Q^{\mathrm{I}-}$ are the known incident flux vectors on the two surfaces of the scatterer due to the incident field. Again, h is the thickness of a pancake-shaped rigid scatterer and in the limit as h goes to zero, equation (10) is degenerate. Like equation (5), equation (10) is ill conditioned when h is small. Equation (10) for $h = 0$ can also be written as

$$\begin{bmatrix} C & 0 \\ 0 & -C \end{bmatrix} \begin{bmatrix} \Delta\Phi \\ \Delta\Phi \end{bmatrix} = \begin{bmatrix} Q^{\mathrm{I}+} \\ Q^{\mathrm{I}-} \end{bmatrix} \qquad (11)$$

which is the discretized form of equation (9).

CBIE/HBIE mixed formulation

So far we have observed some similarities in the properties of the CBIE and the HBIE when the scatterer is thin. The CBIE and HBIE are close to degeneracy for thin scatterers and are degenerate for zero-thickness scatterer. However, there is an important difference in these equations which can be seen from the discretized form. For $h = 0$, unlike equation (5), the terms in each row of (the square matrix) equation (10) add up to zero.

Consider a formulation where the CBIE is used on S^+, equation (3), and the HBIE on S^-, equation (8). Henceforth, we will refer to this formulation as a mixed formulation. The discretized form of the mixed formulation is

$$\begin{bmatrix} \frac{1}{2}I & \sim \frac{1}{2}I \\ \sim -C & C \end{bmatrix} \begin{bmatrix} \Phi^- \\ \Phi^+ \end{bmatrix} = \begin{bmatrix} \Phi^{\mathrm{I}+} \\ Q^{\mathrm{I}-} \end{bmatrix} \qquad (12)$$

where the two rows are the first and second rows of equation (5) and equation (10), respectively. The equations represented by these rows are linearly independent and so the system is well-conditioned for all thicknesses including the zero-thickness scatterer. For $h = 0$, equation (12) can also be written as

$$\begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & -C \end{bmatrix} \begin{bmatrix} \Sigma\Phi \\ \Delta\Phi \end{bmatrix} = \begin{bmatrix} \Phi^{\mathrm{I}+} \\ Q^{\mathrm{I}-} \end{bmatrix} \qquad (13)$$

which is the discretized form of equation (4) on S^+ and of equation (9) on S^-. Unlike equation (12), equation (13) is valid only for a zero-thickness scatterer.

For a thin void problem, one of the several different formulations can give a well-conditioned system of equations. In all of these choices a combination of CBIE and HBIE is required for the exterior domain. For example, if CBIE is used over S^{*+} (a portion of S^+) and HBIE over

$S^+ - S^{*+}$, and if HBIE is used on S^{*-} (the surface opposite to S^{*+} on S^-) and CBIE on $S^- - S^{*-}$, the resulting system of equations is well-behaved. A non-degenerate system can also be obtained if a linear combination of the CBIE and HBIE is used on S^+ and S^-. This can be seen by adding equation (5) and equation (10) which gives

$$\begin{bmatrix} \frac{1}{2}I + C & \sim\frac{1}{2}I - C \\ \sim\frac{1}{2}I - C & \frac{1}{2}I + C \end{bmatrix} \begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix} = \begin{bmatrix} \Phi^{I+} + Q^{I+} \\ \Phi^{I-} + Q^{I-} \end{bmatrix} \tag{14}$$

Although this linear combination requires more computation than equation (12), it does have the advantage of completely eliminating the well-known fictitious frequency difficulty (e.g. [16, 14]). Finally, for a thin-void problem, a formulation obtained by collocating twice at the same points on S^+ (once with CBIE and once with HBIE) gives

$$\begin{bmatrix} \frac{1}{2}I & \sim\frac{1}{2}I \\ C & \sim -C \end{bmatrix} \begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix} = \begin{bmatrix} \Phi^{I+} \\ Q^{I+} \end{bmatrix} \tag{15}$$

and is a well-conditioned system of equations. However, when the void is no longer thin, collocating twice at the same point on S^+ can give poor results.

THE TRANSMISSION PROBLEM

In this section BIEs are written for the scatterer (through which waves propagate) as well as for the surrounding medium. Interface conditions on both ϕ and q need to be stated. The BIEs associated with the exterior domain are the same as those in the previous section except that the normal flux is not known. The CBIE for the exterior domain, equation (1), can be rewritten

$$\frac{1}{2}\phi(\xi_0) + \int_S \frac{\partial G(x,\xi_0)}{\partial n(x)}\phi(x)\,\mathrm{d}S(x) - \int_S G(x,\xi_0)q(x)\,\mathrm{d}S(x) = \phi^I(\xi_0) \tag{16}$$

where S defines the scatterer surface with the normal pointing into the scatterer; ϕ, q and $G(x, \xi)$ are as defined earlier. Also the CBIE for the interior domain can be written

$$\frac{1}{2}\phi^i(\xi_0) - \int_S \frac{\partial G^i(x,\xi_0)}{\partial n(x)}\phi^i(x)\,\mathrm{d}S(x) + \int_S G^i(x,\xi_0)q^i(x)\,\mathrm{d}S(x) = 0 \tag{17}$$

where $G^i(x, \xi) = \exp(ik_i R)/(4\pi R)$, k_i is the wave number for the inclusion, $\phi^i(x)$ and $q^i(x)$ are the potential and normal flux, respectively, on the surface of the inclusion (the normal to the inclusion surface is assumed to point into the inclusion). The potential and the boundary fluxes are related across the boundary according to $\kappa\phi(x) = \phi^i(x)$ and $q(x) = q^i(x)$. For x on S, equation (17) becomes

$$\kappa\frac{1}{2}\phi(\xi_0) - \int_S \frac{\partial G^i(x,\xi_0)}{\partial n(x)}\kappa\phi(x)\,\mathrm{d}S(x) + \int_S G^i(x,\xi_0)q(x)\,\mathrm{d}S(x) = 0 \tag{18}$$

For a given incident field, equations (16) and (18) can be solved for the potential and the flux on the boundary, which then can be used to compute the potential in the far field. Similar formulations have been used to study the scattered field from single and multiple scatterers.

Now consider a thin pancake-shaped inclusion of thickness h with acoustic properties different from the surrounding medium. For this problem the discretized form, neglecting the term of order

h and higher, is

$$\begin{bmatrix} \frac{1}{2}I & \sim\frac{1}{2}I & -B & \sim -B \\ \sim\frac{1}{2}I & \frac{1}{2}I & \sim -B & -B \\ \kappa\frac{1}{2}I & \sim -\kappa\frac{1}{2}I & B^{i} & \sim B^{i} \\ \sim -\kappa\frac{1}{2}I & \kappa\frac{1}{2}I & \sim B^{i} & B^{i} \end{bmatrix} \begin{bmatrix} \Phi^{+} \\ \Phi^{-} \\ Q^{+} \\ Q^{-} \end{bmatrix} = \begin{bmatrix} \Phi^{I+} \\ \Phi^{I-} \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

where I is the identity matrix, B and B^{i} are the matrices associated with the exterior and interior domain, respectively, and Φ, Φ^{I} and Q are as defined earlier. For thin inclusions, the equations represented by first two rows of equation (19), which represent the CBIE for the exterior domain on collocating on the two surfaces, are linearly dependent. As a result, equation (19) is ill conditioned for a thin scatterer. Note that the equations represented by last two rows of equation (19) which correspond to CBIE for the interior domain, unlike the first two rows, are linearly independent.

If one chooses to use the HBIE to solve the inclusion problem, then the integral equation for the exterior domain is

$$\oint_S \frac{\partial^2 G(x,\xi_0)}{\partial n(x)\,\partial n(\xi_0)}\phi(x)\,\mathrm{d}S(x) + \frac{1}{2}q(\xi_0) - \oint_S \frac{\partial G(x,\xi_0)}{\partial n(\xi_0)}q(x)\,\mathrm{d}S(x) = q^{I}(\xi_0) \tag{20}$$

and for the interior domain

$$-\oint_S \frac{\partial^2 G^{i}(x,\xi_0)}{\partial n(x)\,\partial n(\xi_0)}\kappa\phi(x)\,\mathrm{d}S(x) + \frac{1}{2}q(\xi_0) + \oint_S \frac{\partial G^{i}(x,\xi_0)}{\partial n(\xi_0)}q(x)\,\mathrm{d}S(x) = 0 \tag{21}$$

Neglecting terms of order h and higher, equations (20) and (21) in discretized form for a pancake-shaped inclusion become

$$\begin{bmatrix} C & \sim -C & \frac{1}{2}I & \sim -\frac{1}{2}I \\ \sim -C & C & \sim -\frac{1}{2}I & \frac{1}{2}I \\ -\kappa C^{i} & \sim \kappa C^{i} & \frac{1}{2}I & \sim\frac{1}{2}I \\ \sim \kappa C^{i} & -\kappa C^{i} & \sim\frac{1}{2}I & \frac{1}{2}I \end{bmatrix} \begin{bmatrix} \Phi^{+} \\ \Phi^{-} \\ Q^{+} \\ Q^{-} \end{bmatrix} = \begin{bmatrix} \Phi^{I+} \\ \Phi^{I-} \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

The above system of equations is ill conditioned, as well, since the equations represented by the first two rows are linearly dependent and so the HBIE fails to solve the inclusion problem when the inclusion is thin. As in equation (19), the system of equations associated with the interior domain is not the source of the difficulty. In both the cases, equations (19) and (22), the equations associated with the exterior domain, dictate the conditioning of the entire system of equations.

The degeneracy associated with the thin-inclusion problem is the same as that of the thin-void problem. A formulation which uses the CBIE on S^{+} and HBIE on S^{-} of the exterior domain and the CBIE everywhere on the interior domain, in a discretized form for a thin pancake-shaped inclusion, is

$$\begin{bmatrix} \frac{1}{2}I & \sim\frac{1}{2}I & -B & \sim -B \\ \sim -C & C & \sim -\frac{1}{2}I & \frac{1}{2}I \\ \kappa\frac{1}{2}I & \sim -\kappa\frac{1}{2}I & B^{i} & \sim B^{i} \\ \sim -\kappa\frac{1}{2}I & \kappa\frac{1}{2}I & \sim B^{i} & B^{i} \end{bmatrix} \begin{bmatrix} \Phi^{+} \\ \Phi^{-} \\ Q^{+} \\ Q^{-} \end{bmatrix} = \begin{bmatrix} \Phi^{I+} \\ Q^{I-} \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

The above system of equations is well conditioned for all thicknesses.

The special case of an inclusion problem where the inclusion has the same property as the exterior domain is the same as a homogeneous medium. For this special case, $B = B^i$ and equation (23) can be inverted to give

$$\begin{bmatrix} \Phi^+ \\ \Phi^- \\ Q^+ \\ Q^- \end{bmatrix} = \begin{bmatrix} -I & 0 & 0 & I \\ -I & 0 & I & 0 \\ 0 & I & a_{33} & a_{34} \\ 0 & -I & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \Phi^{I+} \\ Q^{I-} \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

As expected, the above equation confirms that the magnitudes of potentials and fluxes on S^+ and S^- are the same as the incident fields. The potentials on these two surfaces are the same while the fluxes are the negatives of each other.

As discussed in the previous section, a number of different formulations for the exterior domain which will guarantee a non-degenerate system of equations can be used. Each of these formulations requires a combination of the CBIE and HBIE as indicated earlier. However, it is important to point out that the CBIE alone or HBIE alone or any combination of these two can be used for the interior domain. For the interior domain, collocation twice on S^+ or S^-, once with CBIE and once with HBIE, will result in an ill-conditioned system of equations. This is unlike that for the exterior domain problem.

The fact that any combination of the CBIE and HBIE on S^+ and S^- for the interior domain may be used without degeneracy, so long as an admissible combination (mixture) of the CBIE and HBIE on S^+ and S^- is used for the exterior domain, requires additional comment. We noted earlier that the mixed problem for an interior thin body, i.e. with ϕ specified on S^+ and q specified on S^- has no degeneracy, even for the thinnest bodies for any combination of CBIE and HBIE. These two situations are similar in the following regards. It is important that the BIEs which characterize either the exterior domain or the interior domain do not degenerate as the thin dimension(s) at issue go to zero. For the exterior domain, this is ensured through the use of the CBIE/HBIE mixed formulas for the domain. For the interior domain, this is ensured, in the inclusion problem, through the interface conditions themselves: whereas for the thin body interior problem, this is ensured (if collocation is done on both S^+ and S^-) only for the mixed problem. The degeneracy or lack thereof can perhaps be most easily seen by examining the linear dependence of groups of equations in the relevant matrices as mentioned above.

When solving scattering problems it is often necessary to address the presence of multiple solutions at a finite number of discrete frequencies, often referred to as fictitious frequencies. One formulation which guarantees a unique solution at all frequencies is that suggested by Burton and Miller [16] where a linear combination of CBIE and HBIE is used. It was shown earlier in equation (14) that a linear combination of the CBIE and HBIE over the entire scatterer surface, Burton and Miller formulation, is non-degenerate for thin voids and scatterers. Thus, it seems that the Burton and Miller formulation can play a duel role: (i) to eliminate fictitious frequencies and (ii) to provide a non-degenerate formulation for thin bodies.

In this work the CBIE and HBIE are interpreted as a CPV and FP, respectively. Such interpretation can be dispensed with if the interior representation is regularized prior to taking the limit of the interior representation to the boundary [17]. The properties of the mixed formulation discussed in this paper as well as the discretized form of the integral equations themselves are the same regardless of the form of the integral equations.

NEAR-SINGULAR INTEGRATION

Here we discuss the near-singular integration problem, difficulty (2) mentioned in the introduction. Singular and near-singular integrals are alike in the sense the integrand varies rapidly over the integration surface. Also, the values of the singular and near-singular integrals are almost the same. Yet, most existing integration techniques are very different for near-singular and singular integration. In this work we revisit a semi-analytical integration technique which has proved to be useful for computation of singular integrals [12, 17] and extend it to near-singular integrals. This technique does not require a high quadrature order and the near-singular integrals can be computed as accurately as all other integrals.

Consider near-singular integrals which appear when the scatterer, in a scattering problem, is thin as in Figure 1. When the collocation point is on S^- and the integral is over S^+, then the near-singular integral

$$I_{ns}(S^+, \xi_0^-) = \int_{S^+} \frac{\partial^2 G(x, \xi_0^-)}{\partial n(x) \partial n(\xi_0^-)} \phi(x) \, dS(x) \tag{25}$$

has to be computed. Note that the integral is over S^+. The integrand of the above integral, with a hypersingular kernel, assumes a large value at ξ_0^+, the point closest to the collocation point on S^+. So subtracting the first two terms of the Taylor's expansion of $\phi(\xi_0)$ about ξ_0^+, the term $|\phi(x) - \phi(\xi_0^+) - \phi_{,p}(\xi_0^+)(x_p - \xi_{0p})| = O(|x - \xi_0|^2)$ and this makes the integrand suited for numerical integration.

$$\begin{aligned} I_{ns}(S^+, \xi_0^-) = & \int_{S^+} \frac{\partial^2 G(x, \xi_0^-)}{\partial n(x) \partial n(\xi_0^-)} [\phi(x) - \phi(\xi_0^+) - \phi_{,p}(\xi_0^+)(x_p - \xi_{0p}^+)] \, dS(x) \\ & + \phi(\xi_0^+) \int_{S^+} \frac{\partial^2 G(x, \xi_0^-)}{\partial n(x) \partial n(\xi_0^-)} \, dS(x) \\ & + \phi_{,p}(\xi_0^+) \int_{S^+} \frac{\partial^2 G(x, \xi_0^-)}{\partial n(x) \partial n(\xi_0^-)} (x_p - \xi_{0p}^+) \, dS(x) \end{aligned} \tag{26}$$

However, the added back terms are just as bad as equation (25) and are not suitable for numerical quadrature. But the last two integrals can be converted into line integrals and other area integrals using Stokes' theorem [12]. These line integrals are non-singular and can be computed using a low quadrature order. The area integrals which arise from the use of Stokes' theorem are of the same or less order of difficulty as those of the first integral in equation (26).

The first integral requires some attention. The integrand, as it appears in the above equation, has a behaviour which is similar to that of a weakly singular integral with the collocation point at ξ_0^+ on S^+. So a polar co-ordinate transformation, as used in the case of weakly singular integrals, will make the first integral of the above equation manageable. However, there is a small neighbourhood about ξ_0^+ where the integrand can rise sharply from zero to a constant value. This region, which is of the same order as $|\xi_0^+ - \xi_0^-|$, can be easily taken care of by dividing the radial direction into two parts, the first with a length equal to $|\xi_0^+ - \xi_0^-|$ from the origin. Additionally, a mapping for the radial direction which shifts the quadrature points close to the origin of the polar co-ordinate system ensures a greater accuracy for the radial integration. A combination of the radial subdivision along with the radial biasing of the Gauss points can completely remove all difficulties associated with the computation of the first integral in equation (26) as well as other integrals which arise from the use of Stokes' theorem.

In the limiting case when the collocation point is on the integration element, the same procedure as discussed above has been used in [12]. Note that the sharp variation of the integrand close to the collocation point becomes a step function, i.e. the radial distance for the function to go from zero to a constant value is zero. Thus, no subdivision of the radial direction or radial biasing of the Gauss points is necessary.

Near-singular integrals with kernels which are strongly singular can also be handled as with the hypersingular kernels but now only the first term of the Taylor's expansion of the boundary variable about the point closest to the collocation point is required [17].

NUMERICAL EXAMPLES

Consider a thin pancake-shaped rigid scatterer as depicted in Figure 2 surrounded by a fluid of indefinite extent. The pancake radius a is unity with thickness $2h$, and it is impinged upon by a plane incident pressure wave with non-dimensional frequency $ka = 1$ at zero incident angle as shown. For ten non-conforming boundary elements (five on each surface), the scattered field at a distance $r = 5a$, for all values of the scattering angle θ, is as shown for three small values of h. The limiting case $h = 0$ is shown for comparison. Convergence of results as h gets smaller is evident, but the gap between the smallest non-zero h and $h = 0$ is due to the fact that the square-root behaviour in ϕ at the rim of the pancake was built into the rim-layer of elements for $h = 0$, whereas, this behaviour was ignored for the other values of h.

Figure 3 depicts a similar set of curves for three small values of h, but here computations are done with the CBIE for comparison. Also, conforming elements are used with the CBIE. The

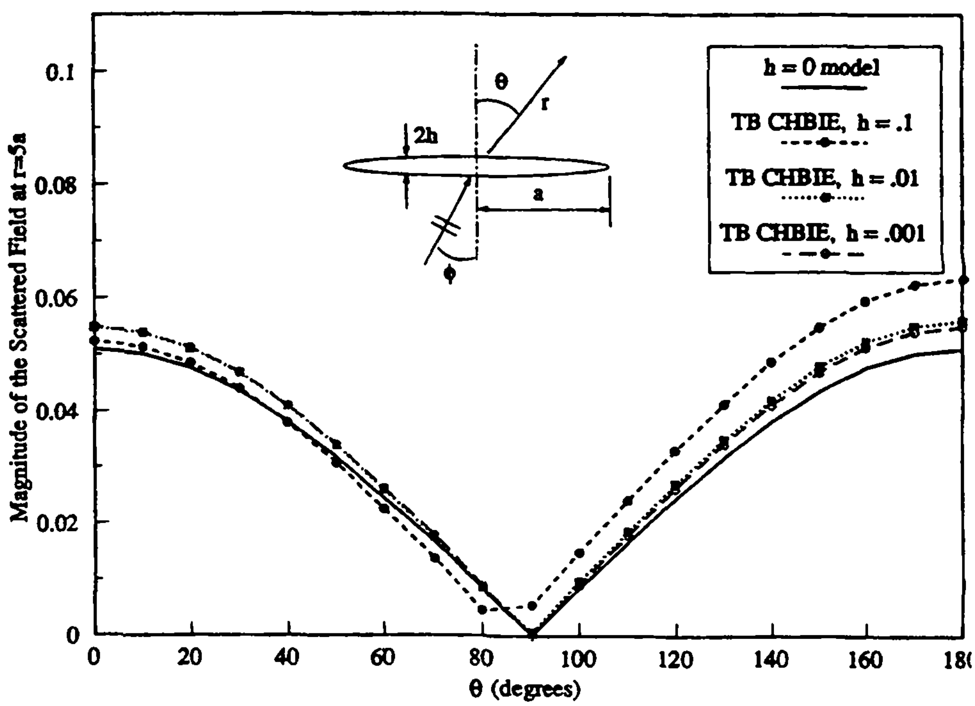

Figure 2. Scattered field, normal incident ($\phi = 0$), $ka = 1$, 10 elements, 80 nodes

inability of the CBIE to deal with the small thickness is evident, even with a relatively large number of elements.

Figure 4 shows a comparison of the $h = 0$ model of the same scatterer with a $h = 0{\cdot}1$ model specifically examining the backscattered field at various distances r. The incident wave is the same as before but at three times the frequency. It is apparent that differences in the two models tend to diminish with increasing r. Results (not shown) for $h = 10^{-7}$ are indistinguishable from the $h = 0$ case.

Figure 5 depicts a comparison similar to that in Figure 4, except that the scatterer is a 'rippled' one and the scattered field is examined in the specular direction. Again $ka = 1$. This scatterer is a preliminary attempt to examine phenomena associated with a corrugated surface or rough crack. Here there is considerable difference with the $h = 0$ model, as expected, especially for incident angles close to hitting the scatterer edge-on, i.e. at 90°, called 'grazing incidence'.

Finally, Figures 6 and 7 depict specular scatter as in Figure 5 except here the data are for non-rigid fluid inclusions in the surrounding fluid with geometry the same as that in Figure 3. The exterior (e) fluid has the same properties in both figures and the (exterior) wave number $k_e a$ is unity in both figures. However, the interior (i) $k_i a$ and the fluid density ratios $\kappa \equiv \rho_e/\rho_i$ are as depicted in each case. Figure 6 shows results for a relatively rigid and massive inclusion, whereas Figure 7 shows for a less rigid and less massive inclusion ($h = 0.1$ in each case). The curves represent data obtained via the CBIE formula for the inclusions as described earlier, but the data points shown were obtained using a lumped model for the inclusion instead of the CBIE. The

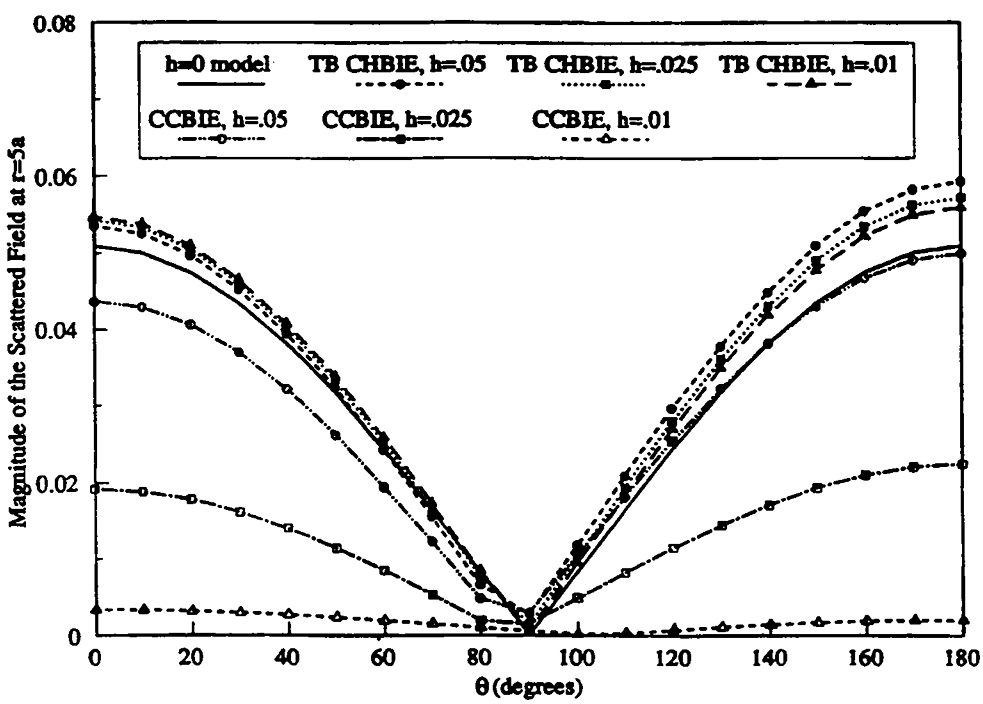

Figure 3. Comparison of the thin-body formulation (TB CHBIE) and the commonly used conventional BIE (CCBIE); normal incident, $ka = 1$, 10 elements for TB CHBIE, 160 (conforming) elements for CCBIE

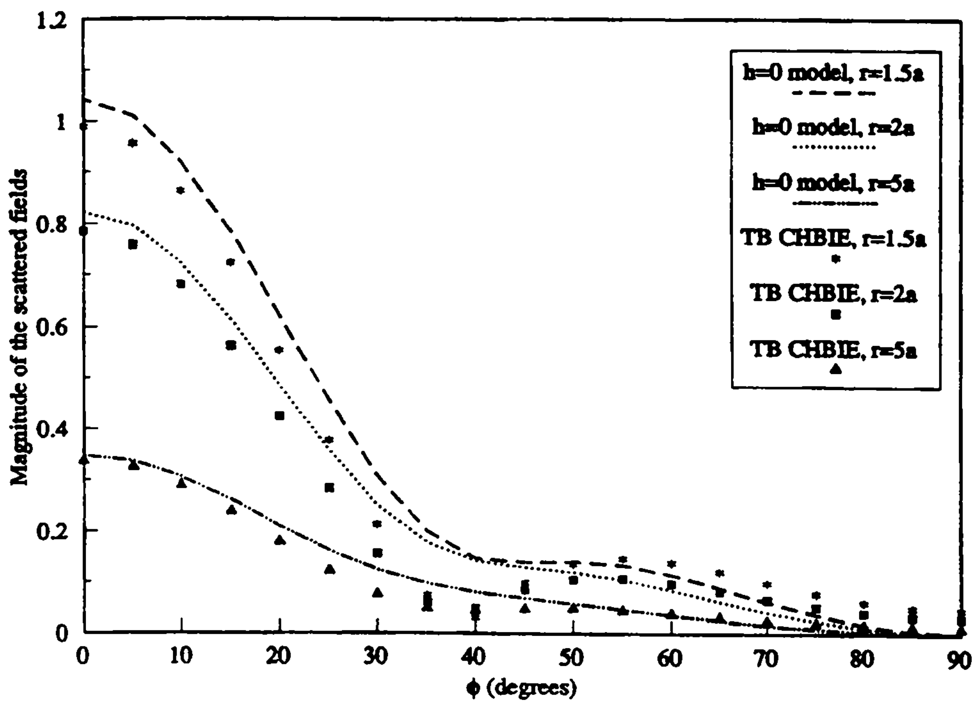

Figure 4. Backscattering, $h = 0{\cdot}1$, $ka = 3$, 40 elements, 320 nodes

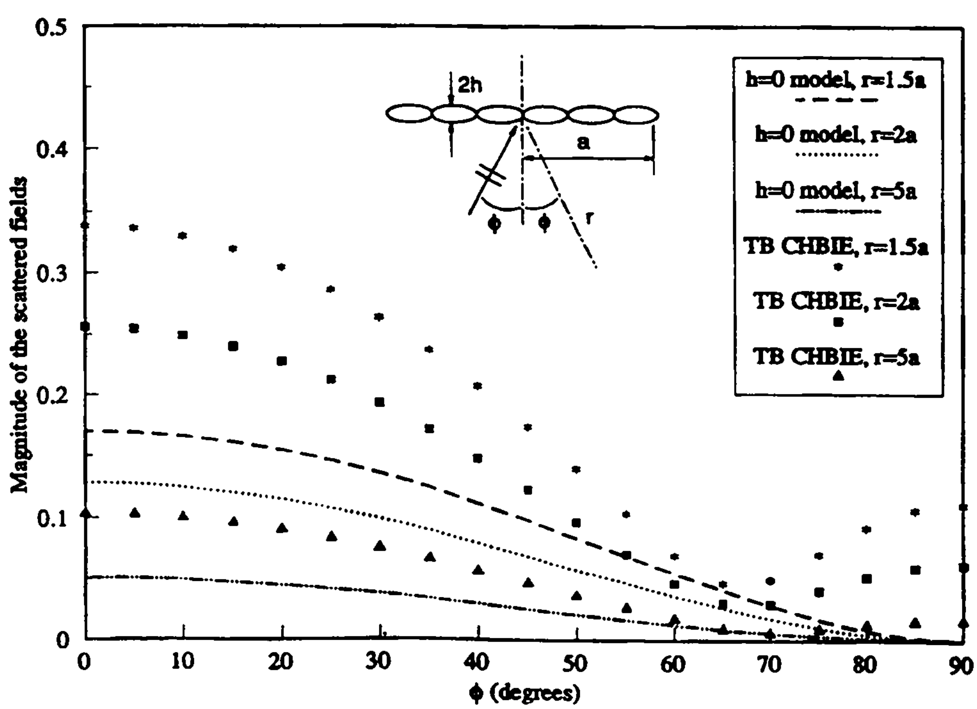

Figure 5. Specular scattering, ripple model, $h = 0{\cdot}3333$, $ka = 1$, 40 elements, 320 nodes

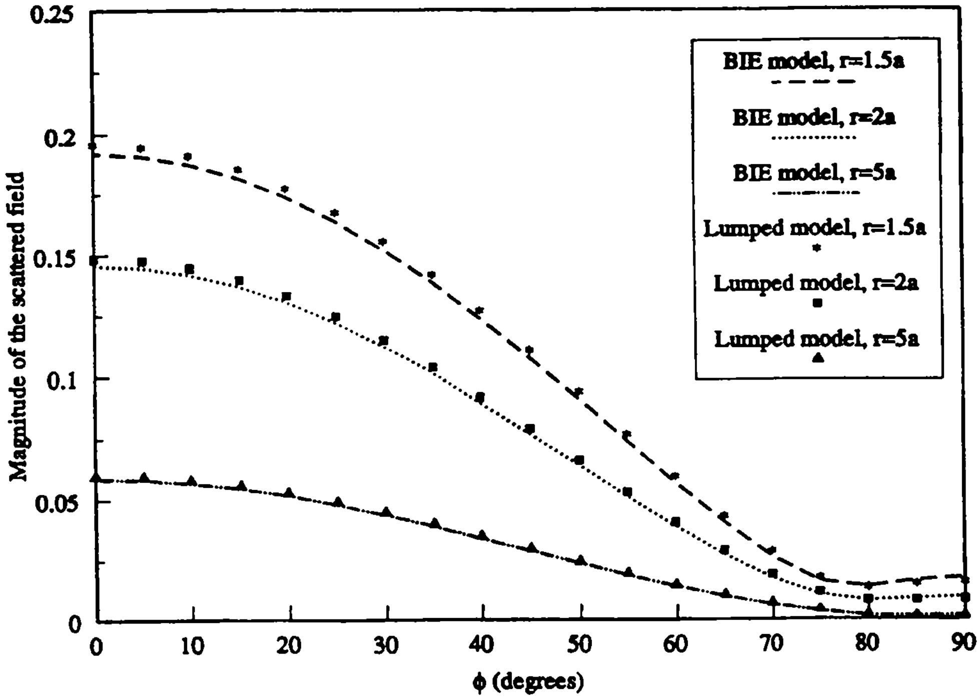

Figure 6. Specular scattering, thin inclusion, $h = 0{\cdot}1$, $k_e a = 1$, $k_i a = 0{\cdot}001$, $\rho_i/\rho_e = 100$

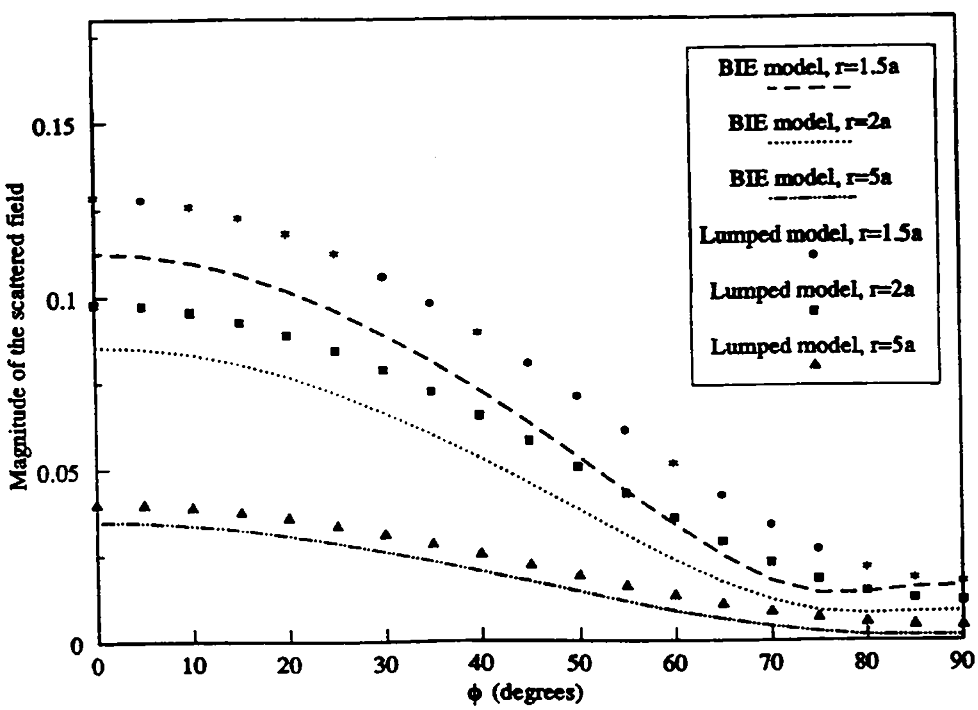

Figure 7. Specular scattering, thin inclusion, $h = 0{\cdot}1$, $k_e a = 1$, $k_i a = 0{\cdot}01$, $\rho_i/\rho_e = 10$

lumped model is given by the relations

$$(q_e^+ + q_e^-) = k_i^2 h \frac{\rho_e}{\rho_i}(\phi_e^+ + \phi_e^-) \tag{27}$$

$$(q_e^+ - q_e^-) = -\frac{1}{h}\frac{\rho_e}{\rho_i}(\phi_e^+ - \phi_e^-) \tag{28}$$

with $q_i = -q_e$ and $\phi_i = (\rho_e/\rho_i)\phi_e$. For parameters chosen in Figure 6, the lumped model works well. However, for the inclusion of Figure 7, the lumped model is apparently less applicable.

ELASTIC-WAVE SCATTERING FROM THIN VOIDS AND INCLUSIONS

The problem of scattering of elastic waves from a thin void in an infinite elastic medium can be formulated as a CBIE (counterpart of equation (1) in elastodynamics)

$$C_{im}(\xi_0)u_m(\xi_0) + \int_S T_{im}(x,\xi_0)u_m(x)\,\mathrm{d}S(x) = \int_S U_{im}(x,\xi_0)t_m(x)\,\mathrm{d}S(x) + u^{\mathrm{I}}(\xi_0) \tag{29}$$

or as a HBIE (counterpart of equation (11) in elastodynamics),

$$D_{im}(\xi_0)t_m(\xi_0) + \int_S V_{im}(x,\xi_0)u_m(x)\,\mathrm{d}S(x) = \int_S W_{im}(x,\xi_0)t_m(x)\,\mathrm{d}S(x) + t^{\mathrm{I}}(\xi_0) \tag{30}$$

where $u = u^{\mathrm{I}} + u^{\mathrm{S}}$ is the total displacement, u^{I} and u^{S} are the incident and the scattered field, respectively, t, t^{I} and t^{S} are the tractions on the boundary corresponding to u, u^{I} and u^{S}, respectively, x and ξ_0 are points on the void surface S. The reader is referred to [17, 18] for the details of the above equations.

As was the case in the scalar problem, the CBIE or the HBIE when used alone to solve thin void (crack) problems results in nearly degenerate (degenerate) system of equations. From a discretized version of the above equations it is easy to show that any of the mixed formulation (as suggested earlier for the scalar problem) for each component of the vector equation will result in a non-degenerate system of equation when the scatterer is a thin void or a crack. Additionally, for a thin-inclusion problem a mixed formulation for the exterior domain is sufficient to result in a non-degenerate system regardless of how the interior domain is modelled.

The singular and near-singular integrals that arise can be regularized, as discussed earlier for the scalar problem, using Taylor's expansion of the density function about the closest point and Stokes' theorem. Again, the singular integration can be seen as a special case of the near-singular integration scheme. The details regarding the use of Stokes' theorem for an arbitrarily shaped surface is discussed in [12].

Implementation of the above mixed formulation and near-singular integration procedure for 3-D vector static and wave problem is in progress.

DISCUSSION

One of the desirable features in a numerical method, especially for problems with 'different geometrical aspects', is the ability of that method to yield accurate results, without requiring too much special attention to those aspects in the form of simplifying assumptions. For example, for elastic behaviour of thin bodies, it would be nice in many applications to get good results from elasticity theory, without having to invoke a simpler plate or shell theory. This is true however more efficient or elegant such a thin-body theory may sometimes be. In this paper we have attempted to show some measures of success in this regard for certain classes of thin-body problems via the BEM.

Clearly, success with the methods in this paper depends on the existence of an exterior medium in contact with the thin bodies. The extent to which the BEM can accomplish anything similar for thin-body (interior) problems alone, or the extent to which such an accomplishment would be meaningful or useful, requires further study.

ACKNOWLEDGEMENTS

Partial support for this work was provided by the National Science Foundation under Grant NSF MSS-8918005. Thanks are due the University of Illinois, Iowa State University and PDA Engineering for additional support. Special thanks are due P. A. Martin, T. J. Rudolphi and S. Nageswaran for several valuable conversations on the structure of thin-body BIE formulations.

REFERENCES

1. T. A. Cruse, *Boundary Element Analysis in Computational Fracture Mechanics*, Kluwer, Boston, 1988.
2. L. J. Gray, 'Boundary element methods for regions with thin internal cavities', *Eng. Anal. with Boundary Elements*, **6**, 180–184 (1989).
3. G. Krishnasamy, F. J. Rizzo and Y. Liu, 'Scattering of acoustic and elastic wave scattering by crack like objects: The role of hypersingular integrals', in D. O. Thompson and D. E. Chimenti (eds.), *Review of progress in Quantitative Nondestructive Evaluation*, Vol. 18, Plenum Press, Maine, 1991.
4. A. Portela and M. H. Aliabadi, 'The dual boundary element method: effective implementation for crack problems', *Int. j. numer. methods eng.*, **33**, 1269–1287 (1992).
5. P. K. Banerjee and R. Butterfield, *Boundary Element Methods in Engineering Science*, McGraw-Hill, London, 1981.
6. F. Hartmann, *Introduction to Boundary Element Theory and Applications*, Springer, New York, 1989.
7. H. R. Kutt, 'The numerical evaluation of principal value integrals by finite-part integration', *Numer. Math.*, **24**, 205–210 (1975).
8. M. P. Brandao, 'Improper integrals in theoretical aerodynamics: the problem revisited', *AIAA J.*, **25**, 1258–1260 (1986).
9. N. Nishimura and S. Kobayashi, 'An improved boundary integral equation method for crack problems.' *Proc. IUTAM Symp.*, San Antonio, TX, April 13–16, 1987, in T. A. Cruse (ed.), *Advanced Boundary Element Method*, Springer, Berlin, Heidelberg, 1987.
10. T. J. Rudolphi, 'The use of simple solutions in the regularization of hypersingular boundary integral equations', *Mathl Comp. Modeling*, **15**, 269–278 (1991).
11. L. J. Gray, L. F. Martha and A. R. Ingraffea, 'Hypersingular integrals in boundary element fracture analysis', *Int. j. numer. methods eng.*, **29**, 1135–1158 (1990).
12. G. Krishnasamy, L. W. Schmerr, T. J. Rudolphi and F. J. Rizzo, 'Hypersingular boundary integral equations: Some applications in acoustic and elastic wave scattering', *J. Appl. Mech.*, **57**, 404–414 (1990).
13. J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Yale University Press, New Haven, CT, 1923.
14. Y. Liu and F. J. Rizzo, 'Non-singular form of the 'hypersingular' boundary integral equation applied to 3-D acoustic wave problems', *Comp. Methods Appl. Mech. Eng.*, **96**, 271–287 (1992).
15. C. C. Chien, H. Rajiyah and S. N. Atluri, 'An effective method for solving the hypersingular integral equations in 3D acoustics', *J. Acoust. Soc. Am.*, **88**, 918–937 (1990).
16. A. J. Burton and G. F. Miller, 'The application of integral equation methods to the numerical solution of some exterior boundary value problems', *Proc. Roy. Soc.* (*London*) **A323**, 201–210 (1980).
17. G. Krishnasamy, F. J. Rizzo and T. J. Rudolphi, 'Hypersingular boundary integral equations: their occurrence, interpretation, regularization and computation', *Developments in Boundary Element Methods, Vol. 7*; Advanced dynamic analysis, Elsevier, Amsterdam, 1992.
18. Y. Liu and F. J. Rizzo, 'Hypersingular boundary integral equations for radiation and scattering of elastic waves in three dimensions', *Comp. Methods. Appl. Mech. Eng.*, **107**, 131–144 (1993).