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On the BEM for acoustic wave problems

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ABSTRACT

The progress of the boundary element method (BEM) for solving acoustic wave problems is reviewed in this paper. The BEM is in a unique position among all the numerical methods available for solving acoustic wave problems. During the last few decades, research on the acoustic BEM has overcome many of the difficulties, and it is now an accurate and efficient numerical method in modeling many large-scale acoustic problems. This paper focuses on reviewing the dual boundary integral equation (BIE) formulation pioneered by Burton and Miller, treatment of the singular integrals involved in the BIEs, discretization considerations, and fast solution methods for solving the acoustic BEM equations. New directions in the research on the acoustic BEM are also discussed, with a few examples to show the potentials of the BEM in modeling aeroacoustics, acoustic metamaterials, bioacoustics, and sound rendering in computer animations.

1. Introduction

Applying the boundary element method (BEM) based on the boundary integral equation (BIE) is a natural way to solve wave propagation problems, including acoustic, elastic and electromagnetic waves. The BEM is especially attractive when the waves propagate in an infinite domain. With the BEM, the discretization with elements happens only on the surfaces of a vibrating structure (like a rotating wind turbine) or a still body (like a dolphin impinged upon by a sonar wave). For infinite or exterior domain problems, the radiation conditions at infinity are satisfied automatically by the BIE. There is no need to truncate the problem domain and implement other type of infinite elements beyond the truncated domain as with other domain based numerical methods. With the advances in the research and development in the last few decades, the BEM for acoustic wave problems has matured and become the preferred method in the numerical tool box for solving acoustic wave problems.

The governing equation for linear time-harmonic acoustic wave problems is the Helmholtz equation, which has been solved by using the BIE and BEM for at least five decades. Some of the early work in this field can be found in Refs. [1–15]. Especially, the work by Burton and Miller in Ref. [2] has been regarded as a classic that has gained the popularity gradually over the year. The Burton–Miller BIE formulation, or the dual BIE formulation, provides a very elegant way to overcome the so-called fictitious frequency difficulties existing in the conventional BIE for exterior acoustic wave problems. Burton and Miller's BIE formulation has been the foundation for many other works on the acoustic BEM (e.g., Refs. [16–34]) and the related singular boundary method (SBM) [35]. In this paper, we will review the dual BIE formulation and highlight some major results in the research on the acoustic BEM.

The paper is organized as follows: In Section 2, we first review the dual BIE formulation for acoustic wave problems. In Section 3, we discuss a few issues associated with the dual BIE formulation. In Section 4, we discuss the fast solution methods for solving the BEM equations. In Section 5, we discuss some future directions of research and show some potential applications of the acoustic BEM in new areas of interests with a few examples. The paper concludes with a summary in Section 6.

2. The dual BIE formulation

Consider an acoustic domain E either inside or outside an enclosed boundary surface S (The case of the latter is shown in Fig. 1). The governing equation in the frequency domain for linear acoustic wave problems is the following Helmholtz equation [36]:

$$\nabla^2\phi + k^2\phi + Q\delta(\mathbf{x},\mathbf{x}_Q) = 0, \qquad \forall \mathbf{x} \in E, \tag{1}$$

where $\phi = \phi(\mathbf{x}, \omega)$ is the complex function representing the acoustic pressure, $Q\delta(\mathbf{x}, \mathbf{x}_Q)$ represents a point source located at $\mathbf{x}_Q$ (inside domain E) with Q representing the intensity of the source and $\delta(\mathbf{x}, \mathbf{x}_Q)$ being the Dirac-δ function, $k = \omega/c$ is the wavenumber with c being the speed of sound in the acoustic medium, and ω is the circular frequency.

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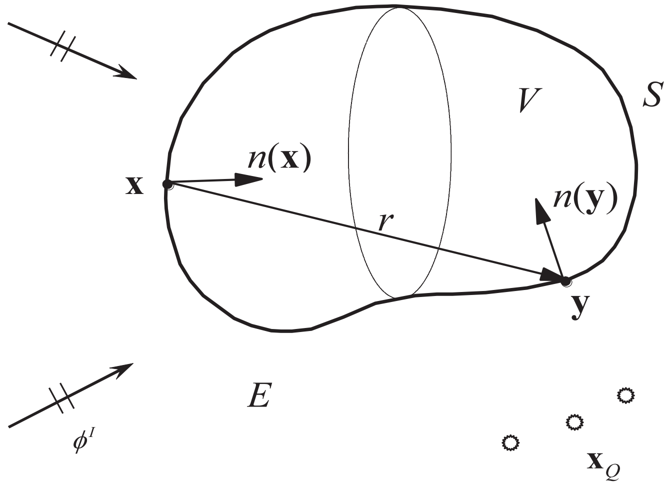

Fig. 1. The acoustic domain E, body V, and boundary S.

The boundary conditions (BCs) for the Helmholtz equation include the following three cases:

(1) Sound pressure is given:

$$\phi = \bar{\phi}, \qquad \forall \mathbf{x} \in S \tag{2}$$

(2) Particle velocity is given:

$$q \equiv \frac{\partial \phi}{\partial n} = \bar{q} = i\omega\rho v_n, \forall \mathbf{x} \in S \tag{3}$$

(3) Impedance of the surface is given:

$$\phi = Z v_n, \forall \mathbf{x} \in S \tag{4}$$

where the overbar indicates a given value, ρ is the mass density of the acoustic medium, v_n is the normal velocity, and Z is the specific acoustic impedance.

For exterior (infinite domain) acoustic wave problems, in addition to the BCs on boundary S, the field at infinity must satisfy the following Sommerfeld radiation condition:

$$\lim_{R\to\infty}\left[R\left|\frac{\partial\phi}{\partial R} - ik\phi\right|\right] = 0, \tag{5}$$

where R is the radius of a large sphere covering the structure and ϕ is either the radiated wave in a radiation problem or the scattered wave in a scattering problem.

Applying the Green's second identity and the fundamental solutions associated with the governing Eq. (1), we can derive the following conventional boundary integral equation (CBIE) formulation [37]:

$$c(\mathbf{x})\phi(\mathbf{x}) = \int_S \left[G(\mathbf{x},\mathbf{y},\omega)q(\mathbf{y}) - F(\mathbf{x},\mathbf{y},\omega)\phi(\mathbf{y})\right] dS(\mathbf{y}) + \phi^I(\mathbf{x}) + QG\left(\mathbf{x},\mathbf{x}_Q,\omega\right), \tag{6}$$

where the constant

$$c(\mathbf{x}) = \begin{cases} 1, & \forall \mathbf{x} \in E, \\ 1/2, & \forall \mathbf{x} \in S (\text{smooth around } \mathbf{x}), \\ 0, & \forall \mathbf{x} \notin E \cup S; \end{cases} \tag{7}$$

$\phi^I(\mathbf{x})$ is the incident wave present for scattering problems, and G and F are the fundamental solutions (kernel functions) for the Helmholtz equation. For example, for 3-D acoustic wave problems, the two kernels are given by:

$$G(\mathbf{x},\mathbf{y},\omega) = \frac{1}{4\pi r}e^{ikr}, \tag{8}$$

$$F(\mathbf{x},\mathbf{y},\omega) \equiv \frac{\partial G(\mathbf{x},\mathbf{y},\omega)}{\partial n(\mathbf{y})} = \frac{1}{4\pi r^2}(ikr-1)r_{,j}n_j(\mathbf{y})e^{ikr}, \tag{9}$$

in which r is the distance from the source point $\mathbf{x}$ to the field point $\mathbf{y}$ (Fig. 1). Eq. (6) (with $\mathbf{x}$ inside domain E) is the representation integral of the solution ϕ inside domain E for both exterior and interior domain problems. Once the values of both ϕ and q are known on S, Eq. (6) can be applied to calculate ϕ everywhere in E, if needed.

The CBIE given in Eq. (6) (with $\mathbf{x}$ on boundary S) can be used to solve for the unknown ϕ and q on S. The integral with the G kernel is a weakly singular integral, whereas the one with the F kernel is a strongly singular integral. A weakly singular form of the CBIE for acoustic wave problems is given by [37–39]:

$$\gamma\phi(\mathbf{x}) + \int_S \left[F(\mathbf{x},\mathbf{y},\omega) - \overline{F}(\mathbf{x},\mathbf{y})\right]\phi(\mathbf{y})dS(\mathbf{y}) + \int_S \overline{F}(\mathbf{x},\mathbf{y})\left[\phi(\mathbf{y}) - \phi(\mathbf{x})\right]dS(\mathbf{y}) = \int_S G(\mathbf{x},\mathbf{y},\omega)q(\mathbf{y})dS(\mathbf{y}) + \phi^I(\mathbf{x}) + QG\left(\mathbf{x},\mathbf{x}_Q,\omega\right), \qquad \forall \mathbf{x} \in S, \tag{10}$$

in which $\overline{F}(\mathbf{x},\mathbf{y}) = F(\mathbf{x},\mathbf{y},0)$ is the static F kernel for potential problems, $\gamma = 0$ for a finite domain and $\gamma = 1$ for an infinite domain. All three integrals in BIE (10) are now at most weakly singular and can be handled readily by numerical integration schemes.

It is well known that the CBIE has two defects. One is that when the CBIE is used for solving exterior acoustic wave problems, the solutions are nonunique at a set of fictitious eigenfrequencies associated with the resonant frequencies of the corresponding interior acoustic problems [2]. This difficulty is referred to as the fictitious eigenfrequency difficulty (FED) [21]. This nonuniqueness is purely a drawback of the mathematical formulation of the problems and does not have any physical significance. Another defect of the CBIE is that when it is used in modeling acoustic wave problems in domains (either interior or exterior) containing thin shapes, the two equations from both sides of the thin shape are identical in the limit as the thickness approaches to zero. This is the so-called thin-shape breakdown (TSB) difficulty [12], which is similar to the difficulty in using the elastostatic CBIE alone in solving crack problems in solids [40,41].

A remedy to both above two difficulties (FED and TSB) in the CBIE is to use the normal derivative BIE combined with the CBIE. Taking the derivative of integral representation (6) with respect to the normal at the point $\mathbf{x}$ and letting $\mathbf{x}$ approach S, we obtain the following so-called hypersingular BIE (HBIE) for acoustic wave problems [21,37]:

$$\tilde{c}(\mathbf{x})q(\mathbf{x}) = \int_S \left[K(\mathbf{x},\mathbf{y},\omega)q(\mathbf{y}) - H(\mathbf{x},\mathbf{y},\omega)\phi(\mathbf{y})\right]dS(\mathbf{y}) + q^I(\mathbf{x}) + QK(\mathbf{x},\mathbf{x}_Q,\omega), \quad \forall \mathbf{x} \in S, \tag{11}$$

where $\tilde{c}(\mathbf{x}) = 1/2$ if S is smooth around $\mathbf{x}$, and $q^I(\mathbf{x})$ is the normal derivative of the incident wave. For 3-D problems, the two new kernels are given by:

$$K(\mathbf{x},\mathbf{y},\omega) \equiv \frac{\partial G(\mathbf{x},\mathbf{y},\omega)}{\partial n(\mathbf{x})} = -\frac{1}{4\pi r^2}(ikr-1)r_{,j}n_j(\mathbf{x})e^{ikr}, \tag{12}$$

$$H(\mathbf{x},\mathbf{y},\omega) \equiv \frac{\partial F(\mathbf{x},\mathbf{y},\omega)}{\partial n(\mathbf{x})} = \frac{1}{4\pi r^3}\left\{(1-ikr)n_j(\mathbf{y}) + \left[k^2r^2 - 3(1-ikr)\right]r_{,j}r_{,l}n_l(\mathbf{y})\right\}n_j(\mathbf{x})e^{ikr}. \tag{13}$$

In HBIE (11), the integral with the kernel K is a strongly singular integral, whereas the one with the H kernel is a hypersingular integral. Similarly, if we introduce the static kernel and a two-term subtraction, and apply the identities satisfied by the static kernels [37–39], we can show that HBIE (11) can be written in the following weakly singular form [21,23]:

$$\gamma q(\mathbf{x}) + \int_S \left[H(\mathbf{x},\mathbf{y},\omega) - \overline{H}(\mathbf{x},\mathbf{y})\right]\phi(\mathbf{y})dS(\mathbf{y}) + \int_S \overline{H}(\mathbf{x},\mathbf{y})\left[\phi(\mathbf{y}) - \phi(\mathbf{x}) - \frac{\partial\phi}{\partial\xi_\alpha}(\mathbf{x})\left(\xi_\alpha - \xi_{o\alpha}\right)\right]dS(\mathbf{y}) + e_{\alpha k}\frac{\partial\phi}{\partial\xi_\alpha}(\mathbf{x})\int_S \left[\overline{K}(\mathbf{x},\mathbf{y})n_k(\mathbf{y}) + \overline{F}(\mathbf{x},\mathbf{y})n_k(\mathbf{x})\right]dS(\mathbf{y})$$

$$= \int_S \left[K(\mathbf{x},\mathbf{y},\omega) + \overline{F}(\mathbf{x},\mathbf{y}) \right] q(\mathbf{y}) dS(\mathbf{y})$$
$$- \int_S \overline{F}(\mathbf{x},\mathbf{y}) \left[q(\mathbf{y}) - q(\mathbf{x}) \right] dS(\mathbf{y}) + q^I(\mathbf{x}) + QK\left(\mathbf{x},\mathbf{x}_Q,\omega\right), \qquad \forall \mathbf{x} \in S, \tag{14}$$

where an overbar indicates the corresponding static kernel, ξ_α ($\alpha = 1$ for 2-D and $\alpha = 1,\ 2$ for 3-D) are local coordinates in the tangential directions at $\mathbf{x} \in S$ and $e_{\alpha k} = \partial \xi_\alpha / \partial x_k$. All the integrals in (14) are now at most weakly singular if ϕ has continuous first derivatives [21,23]. The smoothness requirement on the density functions for the HBIE is discussed in detail in Refs. [42,43] and a possible relaxation of this requirement is discussed in Ref. [23].

It is interesting to note that HBIE (11) also suffers from the fictitious eigenfrequency difficulty when it is used alone in solving exterior acoustic wave problems, nevertheless at a different set of fictitious eigenfrequencies [21]. However, the HBIE can be applied alone in solving thin-shape problems using a single surface BEM model (e.g., scattering from a thin rigid screen in an acoustic medium) [40,44].

A dual BIE formulation given by Burton and Miller in Ref. [2] using a linear combination of CBIE (6) and HBIE (11) can be written symbolically as

$$\text{CBIE} + \beta\ \text{HBIE} = 0, \tag{15}$$

where β is a coupling parameter. This dual BIE formulation is also called CHBIE or composite BIE formulation [21,23] for acoustic wave problems. It was shown by Burton and Miller that the dual BIE in (15) will yield unique solutions at all frequencies, if the value of β is a complex number (i.e., the imaginary part of β is not zero). However, Burton and Miller did not suggest the value or range of values for the coupling parameter β.

Another advantage of the Burton-Miller dual BIE formulation is that it can be applied to model acoustic problems with thin shapes, without the TSB difficulty. It has been shown that there is no degeneracy in the dual BIE formulation, contrary to the case of using the CBIE alone, for modeling wave problems in an elastic domain with thin shapes [45], which is also true for acoustic wave problems. Therefore, the same dual BIE formulation can be applied uniformly to an acoustic BEM problem to provide unique solutions in cases of exterior domain problems and/or domains containing thin shapes, like thin screens, plates or shells, or wind turbine blades. There is no need to switch to a different BIE formulation, such as the single surface BIE or indirect BIE formulations for thin-shape problems.

It should be pointed out that a recent and more elaborate study reported in Ref. [32] by Zheng, et al., shows that the Burton-Miller BIE formulation actually shifts the fictitious eigenfrequencies from the real axis into the complex plane. This can cause new fictitious eigenfrequency difficulties when one start using complex wavenumbers in the acoustic BEM for exterior problems, for which the Burton–Miller dual BIE formulation may fail. However, it appears that for acoustic BEM with real wavenumbers, the Burton-Miller BIE formulation is still the best choice so far for solving exterior acoustic problems. To remove all the fictitious eigenfrequencies from the entire complex plan, a new BIE with double derivatives of the CBIE may need to be employed.

The BIE formulations presented above are based on the frequency domain approach to solving acoustic wave problems, which are adequate for studying harmonic responses of acoustic fields. For time domain acoustic responses, inverse Fourier transform can be applied to obtain the time domain solutions from the frequency domain solutions. However, for many acoustic problems with transient responses, direct time domain BEM will be advantageous, such as in predicting noises due to short impact, squeak, or aerodynamic loads. Regarding the time domain BEM for acoustic problems and discussions on the related issues, some research works can be found in Refs. [46–51].

3. Related issues with the dual BIE formulation

3.1. Choice of the coupling constant β

For the Burton–Miller BIE formulation, it has been suggested that the coupling parameter can be chosen as follows [6,8,17,32]:

$$\beta = \pm i/k, \tag{16}$$

where i is the unit imaginary number, k is the wavenumber, and the selection of plus or minus sign depends on the time factor used [52] and exactly how the CBIE and HBIE terms are added together. This choice has been adopted in most of the acoustic BEM work based on the dual BIE (15). The above expression was proposed based on the analysis and numerical tests using a unit sphere [6,8,17,32].

Another choice of the coupling parameter β can be based on a dimensional analysis. Noting that the HBIE is obtained by taking derivative of the CBIE with respect to a normal direction, which has a length unit, the following choice for the coupling parameter β can be applied:

$$\beta = \pm ih, \tag{17}$$

where h is a typical element size in the BEM mesh. In this way, all terms in the dual BIE formulation (15) will have a consistent unit (acoustic pressure) and more balanced contributions from both the CBIE and HBIE. As the element size should decrease with the increase of the frequency (and thus the wavenumber k), in order to follow the rule of thumb that there should be 6–10 elements per wavelength, the effect of the wavenumber k or frequency is still present implicitly in parameter β as given by (17). This choice of the coupling parameter has been found to yield BEM matrices with better conditioning and more stable results with the dual BIE formulation for complicated domains, as well as simpler ones like a sphere. All numerical examples presented in this paper are based on this selection of parameter β.

3.2. Dealing with the hypersingular integral

Dealing with the so-called hypersingular integral with the H kernel in HBIE (11) has been a troublesome issue in the application of the dual BIE formulation. There are numerous approaches proposed in the literature (see, e.g., Refs. [18,53,54]). Here we emphasize the idea that the best way to deal with the hypersingular integrals is to avoid direct numerical integration whenever possible. First, there are weakly-singular form of the hypersingular BIE as given in Eq. (14). When higher-order elements are used, this form of the HBIE can be used in the discretization [23]. Second, for constant elements, the singular and hypersingular integrals can be evaluated directly [55–57] using either analytical integration or the line integral methods [16,58,59]. Third, the one-term or two-term subtraction techniques can be used locally (on one or a few elements surrounding the collocation point) to regularize the hypersingular integral and then use analytical integration or line integral approach to compute the added back terms [24,44,58]. For constant and linear elements, the added back terms (with the static kernel for potential problems) can be integrated analytically [60,61]. For higher-order elements (quadratic and above), some type of numerical integration will need to be applied even after the one-term or two-term subtractions.

3.3. Choice of the type of elements

Use of the HBIE in the dual BIE formulation introduced a new troublesome issue in the BEM implementation. That is, how to meet the smoothness requirement on the density function [42,43]? Theory requires that the density function of the HBIE should be $C^{1,\alpha}$ continuous locally (i.e., with Holder-continuous first derivatives) in order for the hypersingular integral to converge in the sense of Hadamard finite part [43]. This excludes, in theory, the use of conforming C^0 boundary elements (linear, quadratic, cubic and so on) in the discretization of the HBIE. Use of the C^1 continuous Overhauser elements [20,62,63] was attempted with the dual BIE formulation, as well as the nonconforming quadratic elements [20,21,58] for which the smoothness requirement is met at the collocation point. Both types of elements were found to be difficult to use and not efficient in computation. The use of conforming C^0 quadratic elements [23] was also attempted with the dual BIE formulation, on the premise that the smoothness requirement can be relaxed to piece-wise $C^{1,\alpha}$ continuity. However, it was found that this approach is more troublesome in implementation (for example, the corner problem will show up) and not efficiency in computation.

The best choice of the type of elements to use with the dual BIE formulation seems to be the constant triangular and quadrilateral elements. The constant elements are nonconforming elements, for which the smoothness requirement is satisfied locally. The corner problem is also avoided with constant elements. For wave problems, the concerned data points are often not on the boundary, but away from the boundary where elements are used in the discretization. Therefore, the accuracy of the results at these far-away field points can be achieved using constant elements. It is also relatively easy to implement fast solution methods with constant elements, for which integrals can be either calculated analytically or numerically with some simple quadrature. The use of constant elements with the dual BIE formulation has become a common practice in the research of the acoustic BEM in the last decade, especially with the fast solution methods (see, e.g., Refs. [24,25,27–30]).

4. Fast solution methods

The most important advances in the research on the acoustic BEM in the last three decades are the rapid progresses in solution methods for solving the BEM equations [64]. These so-called fast solution methods include the fast multipole method (FMM), adaptive cross approximation (ACA) method, and the fast direct solvers.

4.1. Fast multipole method

The fast multipole method was pioneered by Rokhlin and Greengard [47,65–67] and has been extended to solving the Helmholtz equation for about three decades (for earlier works, see, e.g., Refs. [24,68–83] and reviews in [47,64]). The main idea of the fast multipole BEM is to apply iterative solvers (such as GMRES [84]) to solve the BEM equations and use the FMM to accelerate the matrix-vector multiplication (the BEM coefficient matrix with the solution vector) in each iteration, without forming the entire coefficient matrix explicitly. A hierarchical tree structure is formed to group all the elements into leaf cells of the tree. For the far-field calculations, the element-to-element interactions in the conventional BEM are replaced with cell-to-cell interactions in the hierarchical tree. The so-called multipole and local expansions of the integrals and some translations are introduced to speed up the summation process. Details of the fast multipole BEM can be found in a tutorial paper [85] or the textbook [37].

Most of the earlier works on the FMM for solving the Helmholtz equation are good for solving acoustic wave problems at either low or high frequencies. For example, in Ref. [69], Rokhlin proposed a diagonal form of the translation matrices for the high-frequency range for the Helmholtz equation. In Ref. [72], Greengard et al. proposed a diagonal translation in the FMM for the low-frequency range. Later, FMM algorithms that integrate the low-frequency and high-frequency expansions and translations and are valid for a wider range of frequencies were proposed in Refs. [82,83]. A comprehensive coverage on solving the Helmholtz equation in 3-D with the FMM can be found in Gumerov and Duraiswami's research volume [81].

Recent advances on the fast multipole BEM for solving the acoustic wave problem has been focused on further improvements of the computational efficiencies of the FMM and kernel independent FMM, including works on adaptive algorithms to further speed up the FMM in acoustic BEM for full-space and half-space problems using the Burton–Miller BIE formulation [24,25], black-box like approaches in solving the Helmholtz equation [86–88], analytical integration of the moments [27] for the high-frequency FMM, fast multipole BEM for acoustic multidomain problems [28] or substructure techniques [89], acoustic shape sensitivity analysis [29], half-space acoustic wave problems with an impedance plane [30], a black-box directional BEM using the Burton-Miller formulation [31], a dual-level fast multipole BEM for acoustic wave problems [34], and a dual-level fast multipole SBM for Laplace and Helmholtz equations [90]. With these recent advances in the last decade or so, it is now possible to solve acoustic BEM models with up-to a few million DOFs (unknowns), in the low to moderately high frequency range (non-dimensional wavenumber up to 100), and within a few hours on a PC.

4.2. Adaptive cross approximation

The adaptive cross approximation method is gaining popularity in the BEM research in recent years due to its kernel independent nature and ease of implementation. ACA also applies iterative linear equation solvers, such as the GMRES. It is based on the concept of the hierarchical matrix ($\mathcal{H}$-matrix) introduced by Hackbusch [91]. When a dense matrix is divided hierarchically into submatrices, some of the submatrices will be low-rank matrices and can be well-approximated using different methods. Based on the theory of the hierarchical matrix, Bebendorf, et al., [92,93] developed the ACA to compute the low-rank matrices and applied the ACA to the BEM. The ACA is fully developed based on the algebra of a BEM matrix, and there is no need to expand the kernel functions. Therefore, the ACA BEM is kernel independent and easier to implement, as compared with the fast multipole BEM. More details of the ACA BEM can be found in Refs. [94,95].

There are only a few research papers on the ACA BEM for solving acoustic wave problems, perhaps due to its ease of implementation. In Ref. [96], Brancati, et al., presented a new ACA BEM to solve 3-D acoustic wave problems. For one example, the new ACA BEM is found to be faster than the fast multipole BEM. The results demonstrated that the ACA BEM can achieve solution efficiency of almost $O(N)$ for low frequency and $O(N\log N)$ for high frequency problems (with N being the number of unknowns). In Ref. [97], Brunner, et al., gave a comparison of the fast multipole BEM and the ACA BEM in solving the Helmholtz equation. They found that the memory use of the fast multipole BEM is less than that of the ACA BEM. However, the implementation and parallelization are much easier for the ACA BEM than for the fast multipole BEM.

4.3. Fast direct solvers

Although the fast multipole BEM and ACA BEM can be very efficient in computation, when the solutions converge with the iterative solver, their efficiencies in solutions can suffer when the acoustic domain is complicated or solutions at higher frequencies are needed. Slow convergence can happen, and the solutions may not converge at all, even when a preconditioner is applied. All these concerns are due to the use of iterative solvers, which can guarantee the convergence of the BEM solutions for many, but not all, practical problems. In addition, the iterative solvers cannot handle multiple right-hand side vectors for a linear system of equations. This is a disadvantage of using the iterative solvers when the effects of multiple input need to be investigated individually, such as in the case of a scattering problem with many different incident waves on a fixed structure. Thus, the research on fast direct solvers is gaining popularity gradually in recent years, especially with the rapid development of the computing hardware with larger memory and faster CPUs.

Research results on the algorithms in the fast direct solvers associated the BEM can be found in Refs. [98–104]. The common idea behind these algorithms is to divide the matrix hierarchically, construct a low-rank approximation for certain submatrices and perform a fast update

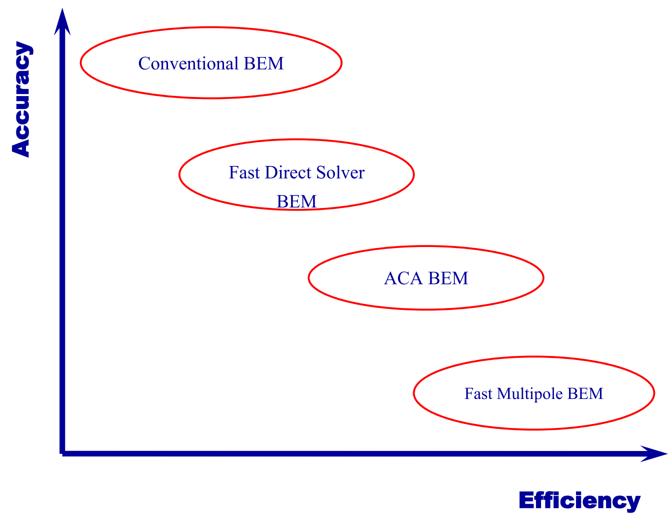

Fig. 2. Accuracy and efficiency with the three fast solution methods.

to the solution recursively, using, for example, the block LU decomposition to invert the matrix, with little or no loss of accuracy in the solution process. A recent implementation of the fast direct solver for the BEM in solving 3-D potential problems is presented in Ref. [105], which can be extended to 3-D acoustic BEM readily.

Based on our experience on the FMM, ACA, and fast direct solver, the FMM is the most efficient method in computation, which can reduce the computational complexity to near $O(N)$ for the BEM regarding both the solution time and memory storage usage. The ACA is the next regarding the efficiency, followed by the fast direct solver. However, the accuracies of the three methods are in the reverse order. A comparison of these methods regarding accuracy vs. efficiency is illustrated in Fig. 2, with the conventional BEM solver as a reference.

5. Prospects for the acoustic BEM

The acoustic BEM has been applied mostly in the areas of structural acoustic or vibro-acoustic type of problems, such as acoustic wave radiation and scattering problems related to machines, cars, airplanes, submarines, wind turbines, and so on. Although still challenging in some cases, the acoustic BEM can now handle these types of analyses routinely with the help of the fast solution methods. The next challenges will be to apply the acoustic BEM to solve acoustic problems with more complex physics and/or in domains with more complicated geometries, such as in aeroacoustics, acoustic metamaterials, bioacoustics, computer animations, and other interdisciplinary areas.

5.1. Aeroacoustics

It is a natural step to extend the acoustic BEM to aeroacoustics, which concerns the sound or noise produced by turbulence and fluid flow over structures. For jet aircraft, high-speed trains, aerial unmanned vehicles (AUVs), space vehicle launch site, and many others, the aeroacoustic noise can be dominant and concerning, in addition to the vibro-acoustic noise. Although CFD (computational fluid dynamics) software tools can be used to compute the aeroacoustic field in the fluid domain theoretically. In practice, it is still very expensive to model aeroacoustic problems with CFD tools alone, especially for large-scale full models like aircraft and high-speed trains in far-field noise predictions. A natural combination will be to apply the CFD and BEM together in modeling large-scale aeroacoustic problems [106]. The CFD can be applied to model the near-field acoustic variables, such as sound pressure and particle velocity, and the BEM is applied to predict the noise radiated to the far field.

For subsonic uniform flow, Wu and Lee proposed a direct BEM for radiation problems in such flow [107]. This work is typical, as the Green's function applied in such cases is similar to the one for the Helmholtz equation, but with the Mach number as a parameter. As the Mach number approaches to zero, the Green's function approaches to the one for the Helmholtz equation. Therefore, many of the results in the BEM for solving the Helmholtz equation can be applied directly to the BEM for solving the sound field at low Mach numbers.

For general aeroacoustics, the governing equation based on the Lighthill analogy [108] and its extensions, including the Ffowcs–Williams Hawkings (FW-H) equation [109], is more involved than the Helmholtz equation. Although in an integral form, the FW-H equation involves a volume integral that is troublesome to deal with and cannot be ignored for flow at high speed. Combined with the CFD, the FW-H equation can be applied to establish models of the noise sources using monopoles, dipoles or quadrupoles, defined at points, on surfaces, or in volumes. Papamoschou, et al., proposed jet noise source models for noise shielding and applied the acoustic fast BEM to calculate the radiated noise at far field [110,111]. Wolf, et al., applied the FMM to accelerate the calculations based on the FW-H integrals [112,113]. They used the hybrid calculation method. The near-field aerodynamic noise source was calculated by CFD, and the far-field aerodynamic noise radiation was calculated by the accelerated FW-H equation. Wolf, et al., also predicted numerically the convective effect of quadrupole aerodynamic noise caused by flow over the NACA0012 airfoil [114]. The results show that the convection effect of airfoil must be considered when performing aerodynamic noise analysis at higher Mach numbers. Mao and Xu also used the spherical harmonic series expansions to accelerate the FW-H equation calculations, and studied the noise prediction for rotating blades [115].

For flow over bluff bodies (with rough surfaces) at high Reynolds numbers, Alomar, et al., proposed a BEM which was validated with experimental results using a cylinder with small circular bumps on the surface [116]. This is an interesting topic as the acoustic BEM can be applied in general to design rough surfaces or coatings over structures in order to reduce the aeroacoustic nose.

The BEM, including the fast BEM, for solving aeroacoustic problems is still in its infancy. The above-mentioned studies on the integral equations and BEM for aeroacoustic problems is still limited to small-scale test problems (such as a simple cylinder, a 2-D wing section model, etc.). For large-scale aeroacoustic problems (such as a full-size aircraft, a group of UAVs, high-speed trains, wind turbines, and other exterior aeroacoustic analyses), aeroacoustic computations with the fast BEM still need to be investigated and further improved.

5.2. Acoustic metamaterials

Modeling acoustic metamaterials presents another interesting and challenging research opportunity for the acoustic BEM. Various metamaterials have been developed in recent years in order to reduce the noise or manipulate the propagation of the acoustic wave in different environment. Metamaterials often have periodic, delicate and small structures that can re-direct the propagation of, or absorb, the acoustic wave in certain frequency ranges called band gaps. Similar to the case of studying fiber-reinforced composites, study of acoustic metamaterials can be done effectively and efficiently using the BEM, because of the complicated geometries of the metamaterial structures and the infinite space in which the wave propagates. These features make the metamaterials perfect candidates for using the BEM to model and optimize such materials. Some of the review articles on acoustic metamaterials can be found in Refs. [117–121].

However, there are not many papers published on the topic of applying the BEM for modeling acoustic metamaterials. Henríquez, et al., studied an acoustic metamaterial model with acoustic losses using both the BEM and FEM, and the BEM and FEM results are compared with existing measurements [122]. In the context of metamaterials concerning

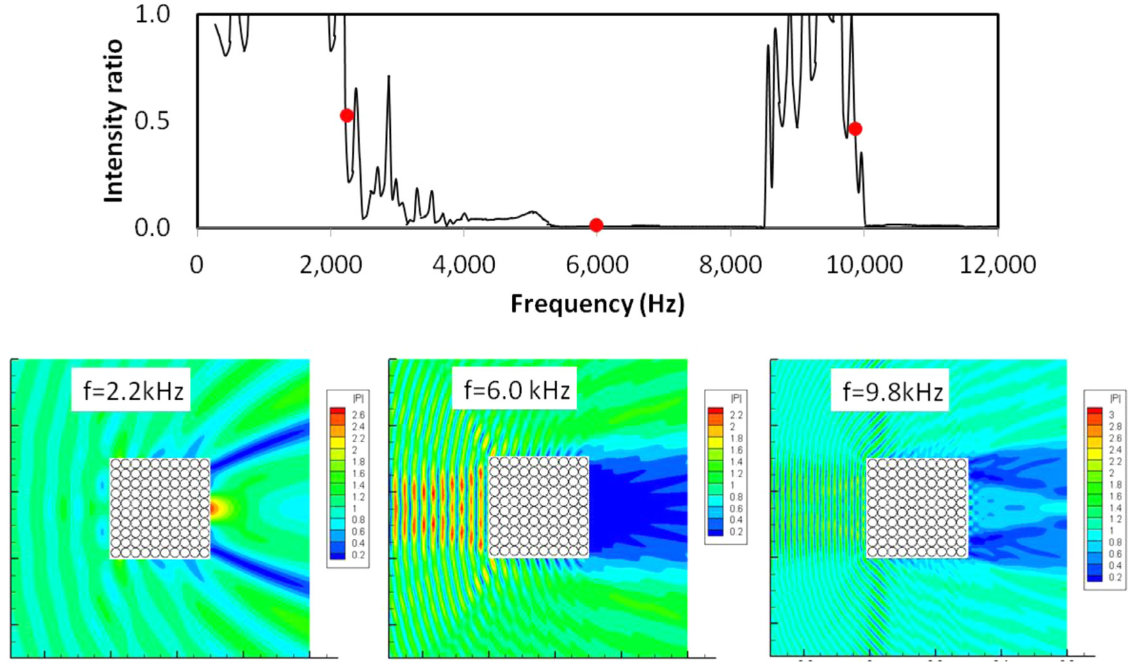

Fig. 3. Contour plots of the sound pressure at three frequencies, with the sound shielding effects of the crystals clearly shown at 6.0 kHz.

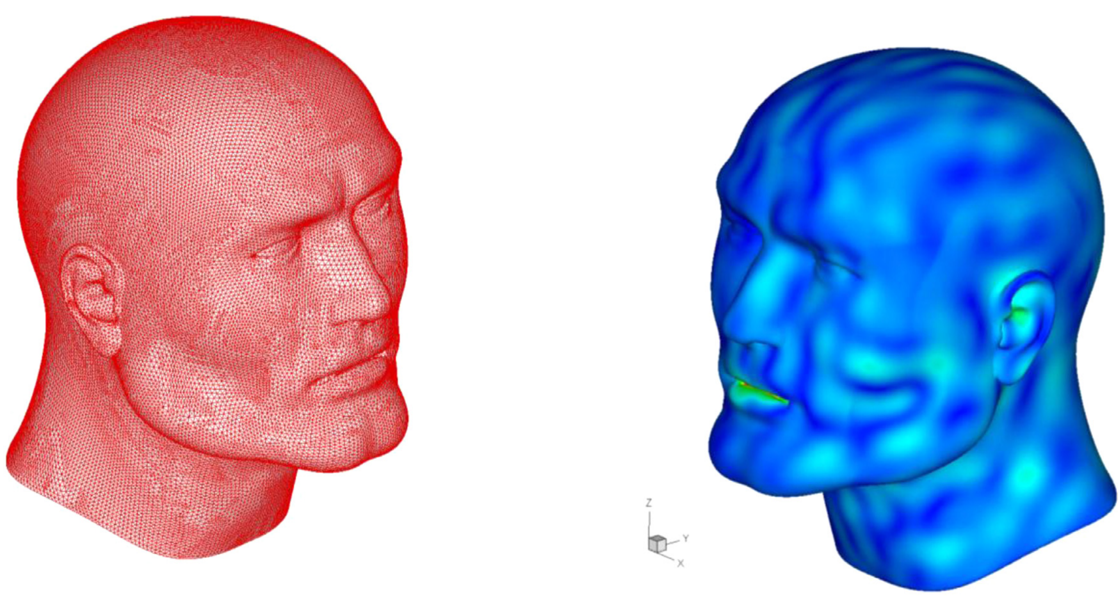

Fig. 4. A human head model: the BEM mesh (left) and the contour plot of sound pressure (right).

elastic waves, Li, et al., applied the elastodynamic BEM on a unknit cell model in studying the band gaps in the metamaterial [123].

In a preliminary study using the example given in Ref. [121], we applied the 2-D acoustic fast multipole BEM to solve the problem of scattering from phononic crystal structures to detect the band gap phenomenon (see Fig. 3). In this example, arrays of rigid and long cylinders are placed in an acoustic medium and impinged upon by an incident wave from left. The sound field around the arrays is computed at different frequencies using the 2-D fast multipole BEM. The plot in Fig. 3 shows the response of the sound pressure (as a ratio of the total wave to the incident wave) at a receiver location (on the right side) vs. the frequency. The plot show clearly a band gap interval (between about 5.5–8.5 kHz) for the model used, which are consistent with the data reported in Ref. [121]. The contour plots in Fig. 3 show the distributions of the sound pressure at three frequencies. The contour plot in the middle for frequency at 6 kHz (within the band gap) clearly shows a "quiet" region on the shadow side the crystals. This preliminary study shows the usefulness and advantages of using the BEM in such research on acoustic metamaterials.

5.3. Bioacoustics

The BEM can be applied to study many bioacoustic problems, such as the navigation of dolphins by means of sonar waves, echoing mechanism of bats using sound waves, hearing loss and hearing devices, and effects of the noise from wind turbine farms to birds or sonar waves to marine animals, and many others. Most of these problems expand in an infinite space and with complicated geometries, which can be modeled most effectively and efficiently by using the BEM. Mey, et al., applied the BEM in studying the acoustic waves scattered by models of sample bats

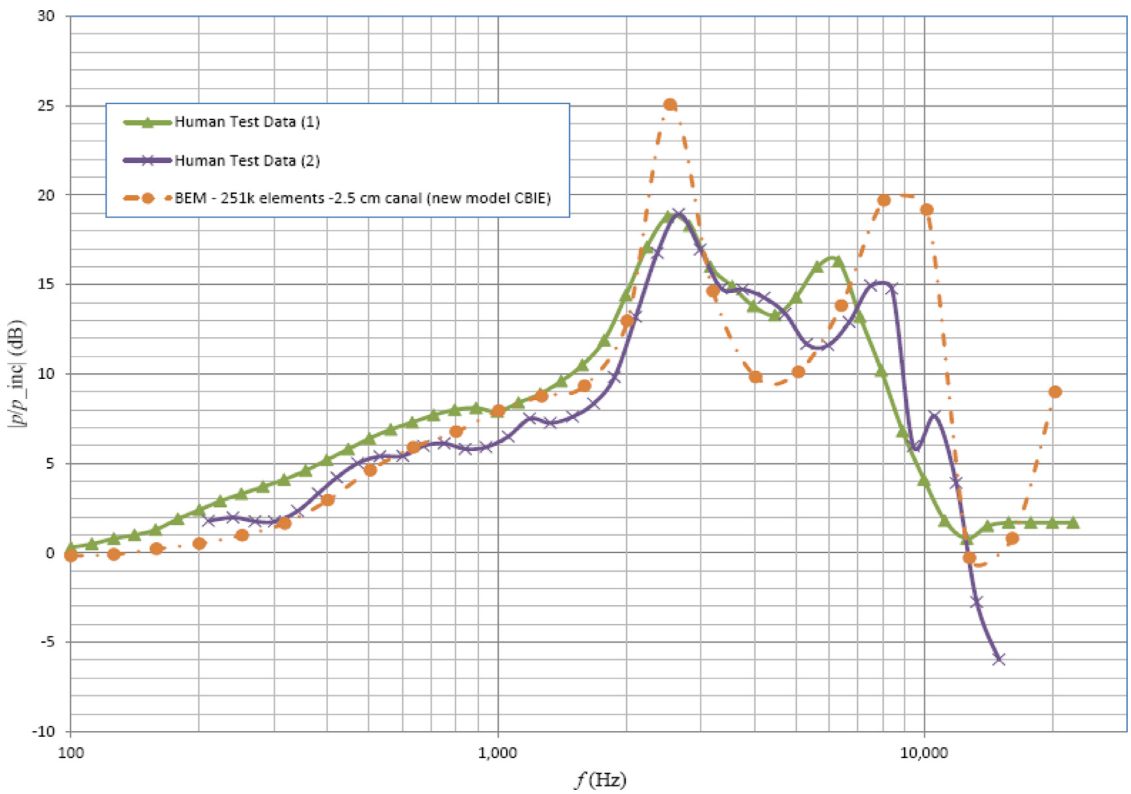

Fig. 5. The response function computed using the fast BEM.

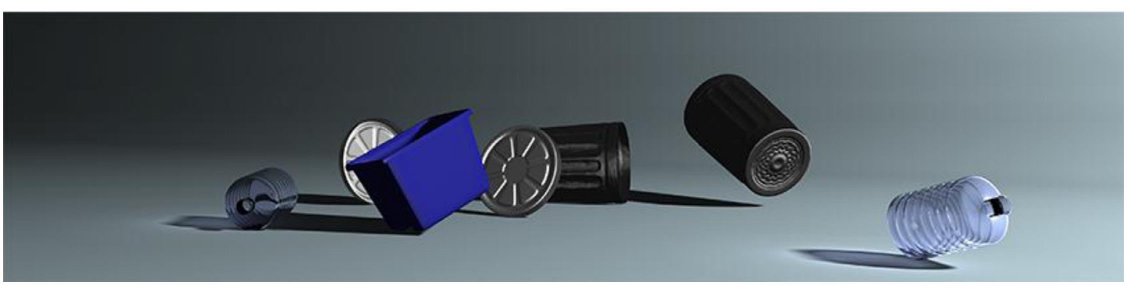

Fig. 6. Sound rendering for computer animation (listen to the computed sound at: https://www.youtube.com/watch?v=cK4wx4pom_0).

and obtained good results in a wide range of frequencies as compared with experiment data [124]. It is argued that the proposed simulation method using the BEM offers distinct advantages over traditional acoustic measurements on real bat specimens.

Fig. 4 shows the BEM mesh and contour plot of sound pressure for a human head model we have studied using the 3-D fast multipole BEM. The head model is impinged upon by an incident wave from the left side, and the frequency response function at the entrance of the canal of the left ear is computed. In Fig. 5, the BEM results are plotted against two curves of measured data from real human samples given in Ref. [125]. The BEM is capable of capturing the trends of the experimental data for frequencies up to 10 kHz, with about 250,000 elements. It is believed that a better match of the BEM results with the experimental data can be achieved, if identical human head models and more accurate impedance boundary conditions of the skin can be applied in the BEM studies.

5.4. Computer animation

The last, but not the least, intriguing and challenging research area for the acoustic BEM is in computer animation for sound rendering [126]. Combined with the computational structural dynamics or rigid-body dynamics software, the acoustic BEM can be applied to compute the sound field from the impact of multiple objects, like in drop tests, fragmentation of a vase, collision of cars, and even falling waters, in a computer animation to replace the costly real sound recordings for those events. The group of Doug James have done some pioneering and very interesting research in this area [127–129], in which the BEM (*FastBEM Acoustics*) was applied in the sound computing part of the simulations. Fig. 5 shows a computer animation of several containers, of different shapes and materials, dropped to the ground. The sound from the impact of the containers with the ground were computed and played back in a video shown at the link (https://www.youtube.com/watch?v=cK4wx4pom_0).

6. Conclusions

With the continued research efforts on the BEM for solving acoustic wave problems, the BEM has emerged as a powerful computational tool, not only in the traditional areas of structural acoustics, but also in a wide range of new application areas. The dual BIE formulation pioneered by Burton and Miller has removed the difficulties of fictitious eigenfrequencies for solving exterior domain problems and for domains containing thin shapes. The realization of the nonsingular nature of the BIE formulations and various analytical integration schemes for acoustic BEM have removed the reservation in implementing and using the BEM for acoustics. The fast solution methods, with the fast multipole method, adaptive cross approximation, and fast direct solvers as the leading choices, have dramatically improved the computational efficiency and expanded the range of applications for the acoustic BEM.

Applications of the acoustic BEM in design of acoustic metamaterials and biomedical devices for sound quality, in sound rendering for computer animation or virtual reality, in virtual testing of acoustic responses to replace costly physical tests, and in integration with CFD methods in aeroacoustics to reduce the noise from high-speed trains, jet aircraft, and unmanned aerial vehicles, will lead the acoustic BEM to an even higher level of importance in computational acoustics. With the continued improvement in computing hardware, further development of the acoustic BEM may prefer formulations, algorithms, solvers and computer codes that are more general, easy to implement, portable, and adapted for high-performance parallel and cloud computing.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.enganabound.2019.07.002.

References

[1] Schenck HA. Improved integral formulation for acoustic radiation problems. J Acoust Soc Am 1968;44:41–58.
[2] Burton AJ, Miller GF. The application of integral equation methods to the numerical solution of some exterior boundary-value problems. Proc R Soc Lond Ser A 1971;323:201–10.
[3] Ursell F. On the exterior problems of acoustics. Proc Camb Philos Soc 1973;74:117–25.
[4] Kleinman RE, Roach GF. Boundary integral equations for the three-dimensional Helmholtz equation. SIAM Rev 1974;16:214–36.
[5] Jones DS. Integral equations for the exterior acoustic problem. Q J Mech Appl Math 1974;27:129–42.
[6] Meyer WL, Bell WA, Zinn BT, Stallybrass MP. Boundary integral solutions of three dimensional acoustic radiation problems. J Sound Vib 1978;59:245–62.
[7] Seybert AF, Soenarko B, Rizzo FJ, Shippy DJ. An advanced computational method for radiation and scattering of acoustic waves in three dimensions. J Acoust Soc Am 1985;77(2):362–8.
[8] Kress R. Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering. Quart J Mech Appl Math 1985;38(2):323–41.
[9] Seybert AF, Rengarajan TK. The use of CHIEF to obtain unique solutions for acoustic radiation using boundary integral equations. J Acoust Soc Am 1987;81:1299–306.
[10] Cunefare KA, Koopmann G. A boundary element method for acoustic radiation valid for all wavenumbers. J Acoust Soc Am 1989;85(1):39–48.
[11] Everstine GC, Henderson FM. Coupled finite element/boundary element approach for fluid structure interaction. J Acoust Soc Am 1990;87(5):1938–47.
[12] Martinez R. The thin-shape breakdown (TSB) of the Helmholtz integral equation. J Acoust Soc Am 1991;90(5):2728–38.
[13] Cunefare KA, Koopmann GH. A boundary element approach to optimization of active noise control sources on three-dimensional structures. J Vib Acoust 1991;113:387–94 No. July.
[14] Ciskowski RD, Brebbia CA. Boundary element methods in acoustics. New York: Kluwer Academic Publishers; 1991.
[15] Wu TW. Boundary element acoustics: fundamentals and computer codes. Southampton: WIT Press; 2000.
[16] Krishnasamy G, Rudolphi TJ, Schmerr LW, Rizzo FJ. Hypersingular boundary integral equations: some applications in acoustic and elastic wave scattering. J Appl Mech 1990;57:404–14.
[17] Amini S. On the choice of the coupling parameter in boundary integral formulations of the exterior acoustic problem. Appl Anal 1990;35:75–92.
[18] Chien CC, Rajiyah H, Atluri SN. An effective method for solving the hypersingular integral equations in 3-D acoustics. J Acoust Soc Am 1990;88(2):918–37.
[19] Wu TW, Seybert AF, Wan GC. On the numerical implementation of a Cauchy principal value integral to insure a unique solution for acoustic radiation and scattering. J Acoust Soc Am 1991;90(1):554–60.
[20] Liu YJ. Development and applications of hypersingular boundary integral equations for 3-D acoustics and elastodynamics. Department of Theoretical and Applied Mechanics, University of Illinois at Urbana-Champaign; 1992.
[21] Liu YJ, Rizzo FJ. A weakly-singular form of the hypersingular boundary integral equation applied to 3-D acoustic wave problems. Comput Methods Appl Mech Eng 1992;96:271–87.
[22] Yang S-A. Acoustic scattering by a hard and soft body across a wide frequency range by the Helmholtz integral equation method. J Acoust Soc Am 1997;102(5):2511–20.
[23] Liu YJ, Chen SH. A new form of the hypersingular boundary integral equation for 3-D acoustics and its implementation with C^0 boundary elements. Comput Methods Appl Mech Eng 1999;173(3-4):375–86.
[24] Shen L, Liu YJ. An adaptive fast multipole boundary element method for three-dimensional acoustic wave problems based on the burton-miller formulation. Comput Mech 2007;40(3):461–72.
[25] Bapat MS, Shen L, Liu YJ. Adaptive fast multipole boundary element method for three-dimensional half-space acoustic wave problems. Eng Anal Bound Elem 2009;33(8-9):1113–23.
[26] Li S, Huang Q. A fast multipole boundary element method based on the improved burton–miller formulation for three-dimensional acoustic problems. Eng Anal Bound Elem 2011;35(5):719–28.
[27] Wu HJ, Liu YJ, Jiang WK. Analytical integration of the moments in the diagonal form fast multipole boundary element method for 3-D acoustic wave problems. Eng Anal Bound Elem 2012;36(2):248–54.
[28] Wu HJ, Liu YJ, Jiang WK. A fast multipole boundary element method for 3D multi--domain acoustic scattering problems based on the burton–miller formulation. Eng Anal Bound Elem 2012;36(5):779–88.
[29] Zheng C, Matsumoto T, Takahashi T, Chen H. A wideband fast multipole boundary element method for three dimensional acoustic shape sensitivity analysis based on direct differentiation method. Eng Anal Bound Elem 2012;36(3):361–71.
[30] Wu H, Liu YJ, Jiang W, Lu W. A fast multipole boundary element method for three--dimensional half-space acoustic wave problems over an impedance plane. Int J Comput Methods 2015;12(1):1350090.
[31] Cao Y, Wen L, Xiao J, Liu YJ. A fast directional BEM for large-scale acoustic problems based on the Burton–Miller formulation. Eng Anal Bound Elem 2015;50:47–58.
[32] Zheng C-J, Chen H-B, Gao H-F, Du L. Is the Burton–Miller formulation really free of fictitious eigenfrequencies? Eng Anal Bound Elem 2015;59:43–51.
[33] Wu H, Ye W, Jiang W. A collocation BEM for 3D acoustic problems based on a non-singular Burton–Miller formulation with linear continuous elements. Comput Methods Appl Mech Eng 2018;332:191–216.
[34] Li J, Chen W, Qin Q. A modified dual-level fast multipole boundary element method based on the burton–miller formulation for large-scale three-dimensional sound field analysis. Comput Methods Appl Mech Eng 2018;340:121–46.
[35] Fu Z-J, Chen W, Gu Y. Burton–Miller-type singular boundary method for acoustic radiation and scattering. J Sound Vib 2014;333(16):3776–93.
[36] Pierce AD. Acoustics: an introduction to its physical principles and applications. New York: ASA; 1991.
[37] Liu YJ. Fast multipole boundary element method - Theory and applications in engineering. Cambridge: Cambridge University Press; 2009.
[38] Liu YJ, Rudolphi TJ. Some identities for fundamental solutions and their applications to weakly-singular boundary element formulations. Eng Anal Bound Elem 1991;8(6):301–11.
[39] Liu YJ, Rudolphi TJ. New identities for fundamental solutions and their applications to non-singular boundary element formulations. Comput Mech 1999;24(4):286–92.
[40] Krishnasamy G, Rizzo FJ, Liu YJ. Boundary integral equations for thin bodies. Int J Numer Methods Eng 1994;37:107–21.
[41] Liu YJ. Analysis of shell-like structures by the boundary element method based on 3-D elasticity: formulation and verification. Int J Numer Methods Eng 1998;41:541–58.
[42] Krishnasamy G, Rizzo FJ, Rudolphi TJ. Continuity requirements for density functions in the boundary integral equation method. Comput Mech 1992;9:267–84.
[43] Martin PA, Rizzo FJ. Hypersingular integrals: how smooth must the density be? Int J Numer Methods Eng 1996;39:687–704.
[44] Krishnasamy G, Rizzo FJ, Liu YJ. Some advances in boundary integral methods for wave-scattering from cracks. Acta Mechanica, 3; 1992. Springer p. 55–65.
[45] Liu YJ, Rizzo FJ. Scattering of elastic waves from thin shapes in three dimensions using the composite boundary integral equation formulation. J Acoust Soc Am 1997;102(2):926–32 No. Pt.1, August.
[46] Schanz M, Antes H. Application of 'operational quadrature methods' in time domain boundary element methods. Mechanica 1997;32(3):179–86.
[47] Nishimura N. Fast multipole accelerated boundary integral equation methods. Appl Mech Rev 2002;55(4):299–324 (July).
[48] Langer S, Schanz M. Time domain boundary element method. In: Marburg S, Nolte B, editors. Computational acoustics of noise propagation in fluids – finite and boundary element methods. New York: Springer; 2008. p. 495–516.
[49] Schanz M, Ruberg T, Kielhorn L. Time domain BEM: numerical aspects of collocation and Galerkin formulations. Recent advances in boundary element Methods: a volume to honor professor Dimitri Beskos. Manolis GD, Polyzos D, editors. Springer; 2009.

[50] Gu X, Li S. Convolution quadrature time-domain boundary element method for two-dimensional aeroacoustic noise prediction. Acoust Phys 2018;64(6):731–41.
[51] Gimperlein H, Oezdemir C, Stephan EP. Time domain boundary element methods for the neumann problem: error estimates and acoustic problems. J Comput Math 2018;36(1):70–89.
[52] Marburg S. The Burton and Miller method: unlocking another mystery of its coupling parameter. J Comput Acoust 2016;24(01):1550016.
[53] Sladek V, Sladek J, Tanaka M. Regularization of hypersingular and nearly singular integrals in the potential theory and elasticity. Int J Numer Methods Eng 1993;36:1609–28.
[54] Chen JT, Hong HK. Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series. Appl Mech Rev 1999;52(1):17–33.
[55] Matsumoto T, Zheng C, Harada S, Takahashi T. Explicit evaluation of hypersingular boundary integral equation for 3-D Helmholtz equation discretized with constant triangular element. J Comput Sci Technol 2010;4(3):194–206.
[56] Zheng C, Matsumoto T, Takahashi T, Chen H. Explicit evaluation of hypersingular boundary integral equations for acoustic sensitivity analysis based on direct differentiation method. Eng Anal Bound Elem 2011;35(11):1225–35.
[57] Tadeu A, António J. 3D acoustic wave simulation using BEM formulations: closed form integration of singular and hypersingular integrals. Eng Anal Bound Elem 2012;36(9):1389–96.
[58] Krishnasamy G, Rizzo FJ, Liu YJ. Scattering of acoustic and elastic waves by crack-like objects: the role of hypersingular integral equations. Review of progress in quantitative nondestructive evaluation. Thompson DO, Chimenti DE, editors. Brunswick, Maine: Plenum Press; 1991.
[59] Krishnasamy G, Rizzo FJ, Rudolphi TJ, et al. Hypersingular boundary integral equations: their occurrence, interpretation, regularization and computation Developments in boundary element methods. Banerjee and PK, et al., editors, London: Elsevier Applied Science Publishers; 1991. Chapter 7.
[60] Fukui T. Research on the boundary element method – development and applications of fast and accurate computations. Department of Global Environment Engineering, Kyoto University; 1998. Ph.D. Dissertation (in Japanese).
[61] Salvadori A. Analytical integrations of hypersingular kernel in 3D BEM problems. Comp Methods Appl Mech Eng 2001;190:3957–75.
[62] Gray LJ, Soucie CS. A hermite interpolation algorithm for hypersingular boundary integrals. Int J Numer Methods Eng 1993;36:2357–67.
[63] editors Liu YJ, Rizzo FJ. Application of overhauser $C^{(1)}$ continuous boundary elements to 'hypersingular' BIE for 3-D acoustic wave problems. In: Brebbia CA, Gipson GS, editors. Boundary elements XIII Computation Mechanics Publications; 1991. Tulsa, OK p. 957–66.
[64] Liu YJ, Mukherjee S, Nishimura N, Schanz M, Ye W, Sutradhar A, Pan E, Dumont NA, Frangi A, Saez A. Recent advances and emerging applications of the boundary element method. Appl Mech Rev 2012;64 No. March, 1-38.
[65] Rokhlin V. Rapid solution of integral equations of classical potential theory. J Comp Phys 1985;60:187–207.
[66] Greengard LF, Rokhlin V. A fast algorithm for particle simulations. J Comput Phys 1987;73(2):325–48.
[67] Greengard LF. The rapid evaluation of potential fields in particle systems. Cambridge: The MIT Press; 1988.
[68] Rokhlin V. Rapid solution of integral equations of scattering theory in two dimensions. J Comput Phys 1990;86(2):414–39.
[69] Rokhlin V. Diagonal forms of translation operators for the helmholtz equation in three dimensions. Appl Comput Harmon Anal 1993;1(1):82–93.
[70] Epton M, Dembart B. Multipole translation theory for the three dimensional laplace and helmholtz equations. SIAM J Sci Comput 1995;16:865–97.
[71] Koc S, Chew WC. Calculation of acoustical scattering from a cluster of scatterers. J Acoust Soc Am 1998;103(2):721–34.
[72] Greengard LF, Huang J, Rokhlin V, Wandzura S. Accelerating fast multipole methods for the Helmholtz equation at low frequencies. IEEE Comput Sci Eng 1998;5(3):32–8.
[73] Tournour MA, Atalla N. Efficient evaluation of the acoustic radiation using multipole expansion. Int J Numer Methods Eng 1999;46(6):825–37.
[74] Darve E. The fast multipole method I: error analysis and asymptotic complexity. SIAM J Numer Anal 2000;38(1):98–128.
[75] Darve E. The fast multipole method: numerical implementation. J Comput Phys 2000;160(1):195–240.
[76] Yasuda Y, Sakuma T. A technique for plane-symmetric sound field analysis in the fast multipole boundary element method. J Comput Acoust 2005;13(1):71–85.
[77] Gumerov NA, Duraiswami R. Recursions for the computation of multipole translation and rotation coefficients for the 3-D helmholtz equation. SIAM J Sci Comput 2003;25(4):1344–81.
[78] Darve E, Havé P. Efficient fast multipole method for low-frequency scattering. J Comput Phys 2004;197(1):341–63.
[79] Fischer M, Gauger U, Gaul L. A multipole galerkin boundary element method for acoustics. Eng Anal Bound Elem 2004;28:155–62.
[80] Chen JT, Chen KH. Applications of the dual integral formulation in conjunction with fast multipole method in large-scale problems for 2D exterior acoustics. Eng Anal Bound Elem 2004;28(6):685–709.
[81] Gumerov NA, Duraiswami R. Fast multipole methods for the helmholtz equation in three dimensions. Amsterdam: Elsevier; 2004.
[82] Cheng H, Crutchfield WY, Gimbutas Z, Greengard LF, Ethridge JF, Huang J, Rokhlin V, Yarvin N, Zhao J. A wideband fast multipole method for the helmholtz equation in three dimensions. J Comput Phys 2006;216(1):300–25.
[83] Gumerov NA, Duraiswami R. A broadband fast multipole accelerated boundary element method for the three dimensional helmholtz equation. J Acoust Soc Am 2009;125(1):191–205.
[84] Saad Y, Schultz M. A generalized minimal residual algorithm for solving nonsymmetric linear system. SIAM J Stat Comput 1986;7:856–69.
[85] Liu YJ, Nishimura N. The fast multipole boundary element method for potential problems: a tutorial. Eng Anal Bound Elem 2006;30(5):371–81.
[86] Ying L, Biros G, Zorin D. A kernel-independent adaptive fast multipole algorithm in two and three dimensions. J Comput Phys 2004;196(2):591–626.
[87] Engquist B, Ying L. Fast directional algorithms for the helmholtz kernel. J Comput Appl Math 2010;234(6):1851–9.
[88] Fong W, Darve E. The black-box fast multipole method. J Comput Phys 2009;228(23):8712–25.
[89] Cui XB, Ji ZL. Fast multipole boundary element approaches for acoustic attenuation prediction of reactive silencers. Eng Anal Bound Elem 2012;36(7):1053–61.
[90] Li J, Chen W, Fu Z. A modified dual-level algorithm for large-scale three-dimensional laplace and helmholtz equation. Comput Mech 2018;62(4):893–907.
[91] Hackbusch W. A sparse matrix arithmetic based on $\mathcal{H}$-matrices. Part I: introduction to $\mathcal{H}$-matrices. Computing 1999;62(2):89–108.
[92] Bebendorf M. Approximation of boundary element matrices. Numer Math 2000;86(4):565–89.
[93] Bebendorf M, Rjasanow S. Adaptive low-rank approximation of collocation matrices. Computing 2003;70:1–24.
[94] Rjasanow S, Steinbach O. The fast solution of boundary integral equations. Berlin: Springer; 2007.
[95] Bebendorf M. Hierarchical Matrices: a means to efficiently solve elliptic boundary value problems. Berlin: Springer-Verlag; 2008.
[96] Brancati A, Aliabadi MH, Benedetti I. Hierarchical adaptive cross approximation GMRES technique for solution of acoustic problems using the boundary element method. CMES Comput Model Eng Sci 2009;43(2):149–72.
[97] Brunner D, Junge M, Rapp P, Bebendorf M, Gaul L. Comparison of the fast multipole method with hierarchical matrices for the Helmholtz-BEM. CMES Comput Model Eng Sci 2010;58(2):131–58.
[98] Martinsson PG, Rokhlin V. A fast direct solver for boundary integral equations in two dimensions. J Comput Phys 2005;205(1):1–23.
[99] Greengard L, Gueyffier D, Martinsson P-G, Rokhlin V. Fast direct solvers for integral equations in complex three-dimensional domains. Acta Numer 2009;18:243–75.
[100] Ho K, Greengard L. A fast direct solver for structured linear systems by recursive skeletonization. SIAM J Sci Comput 2012;34(5):A2507–32.
[101] Ambikasaran S, Darve E. An O(NlogN) fast direct solver for partial hierarchically semi-separable matrices. J Sci Comput 2013;57(3):477–501.
[102] Lai J, Ambikasaran S, Greengard L. A fast direct solver for high frequency scattering from a large cavity in two dimensions. SIAM J Sci Comput 2014;36(6):887–903.
[103] Corona E, Martinsson P-G, Zorin D. An O(N) direct solver for integral equations on the plane. Appl Comput Harmon Anal 2015;38(2):284–317.
[104] Coulier P, Pouransari H, Darve E. The inverse fast multipole method: using a fast approximate direct solver as a preconditioner for dense linear systems. SIAM J Sci Comput 2015;39(3):761–96.
[105] Huang S, Liu YJ. A new fast direct solver for the boundary element method. Comput Mech 2017;60(3):379–92.
[106] Tosh A, Liever P, Owens F, Liu YJ. A high-fidelity CFD/BEM methodology for launch pad acoustic environment prediction. In: Proceedings of the 18th AIAA/CEAS aeroacoustics conference (33rd AIAA aeroacoustics conference). Colorado Springs: AIAA; 2012. Paper Number: AIAA 2012-2107.
[107] Wu TW, Lee L. A direct boundary integral formulation for acoustic radiation in a subsonic uniform flow. J Sound Vib 1994;175(1):51–63.
[108] Lighthill MJ. On sound generated aerodynamically. I. General theory. Proc R Soc Lond A Math Phys Sci 1952;211(1107):564–87.
[109] Ffowcs-Williams JE, Hawkings DL. Sound generation by turbulence and surfaces in arbitrary motion. Philos Trans R Soc Lond Ser A Math Phys Sci 1969;264(1151):321–42.
[110] Papamoschou D. Prediction of jet noise shielding. In: Proceedings of the 48th AIAA aerospace sciences meeting including the new horizons forum and aerospace exposition. American Institute of Aeronautics and Astronautics; 2010.
[111] Papamoschou D, Mayoral S. Modeling of jet noise sources and their diffraction with uniform flow. In: Proceedings of the 51st AIAA aerospace sciences meeting including the new horizons forum and aerospace exposition. American Institute of Aeronautics and Astronautics; 2013.
[112] Wolf WR, Lele SK. Acoustic analogy formulations accelerated by fast multipole method for two-dimensional aeroacoustic problems. AIAA J 2010;48(10):2274–85.
[113] Wolf WR, Lele SK. Aeroacoustic integrals accelerated by fast multipole method. AIAA J 2011;49(7):1466–77.
[114] Wolf WR, Azevedo JLF, Lele SK. Convective effects and the role of quadrupole sources for aerofoil aeroacoustics. J Fluid Mech 2012;708:502–38.
[115] Mao Y, Xu C. Accelerated method for predicting acoustic far field and acoustic power of rotating source. AIAA J 2016;54(2):603–15.
[116] Alomar A, Angland D, Zhang X. Extension of roughness noise to bluff bodies using the boundary element method. J Sound Vib 2018;414:318–37.
[117] Gorishnyy T, Maldovan M, Ullal C, Thomas E. Sound ideas. Phys World 2005;12:1–6.
[118] Liang Z, Li J. Extreme acoustic metamaterial by coiling up space. Phys Rev Lett 2012;108(11):114301.
[119] Ma G, Sheng P. Acoustic metamaterials: from local resonances to broad horizons. Sci Adv 2016;2(2):e1501595.
[120] Cummer SA, Christensen J, Alù A. Controlling sound with acoustic metamaterials. Nat Rev Mater 2016;1:16001.
[121] Pennec Y, Vasseur JO, Djafari-Rouhani B, Dobrzyński L, Deymier PA. Two-dimensional phononic crystals: examples and applications. Surf Sci Rep 2010;65(8):229–91.

[122] Henríquez VC, Andersen PR, Jensen JS, Juhl PM, Sánchez-Dehesa J. A numerical model of an acoustic metamaterial using the boundary element method including viscous and thermal losses. J Comput Acoust 2017;25(04):1750006.
[123] Li F-L, Wang Y-S, Zhang C, Yu G-L. Boundary element method for band gap calculations of two-dimensional solid phononic crystals. Eng Anal Bound Elem 2013;37(2):225–35.
[124] Mey FD, Reijniers J, Peremans H, Otani M, Firzlaff U. Simulated head related transfer function of the phyllostomid bat phyllostomus discolor. J. Acoust. Soc. Am. 2008;124(4):2123–32.
[125] Mehrgardt S, Mellert V. Transformation characteristics of the external human ear. J Acoust Soc Am 1977;61(6):1567–76.
[126] Wang J-H, Qu A, Langlois TR, James DL. Toward wave-based sound synthesis for computer animation. ACM TransGraph 2018;37(4):1–16 SIGGRAPH 2018.
[127] Chadwick J, An S, James DL. Harmonic shells: a practical nonlinear sound model for near-rigid thin shells. ACM Trans Graph 2009;28(5):119:1–119:10 SIGGRAPH 2009.
[128] Zheng C, James DL. Rigid-body fracture sound with precomputed soundbanks. ACM Trans Graph 2010;29(3):1–13 SIGGRAPH 2010.
[129] Zheng C, James DL. Toward high-quality modal contact sound. ACM Trans Graph 2011;30(4):1–11 SIGGRAPH 2011.