An Introduction to the **Boundary Element Method (BEM)** *and Its Applications in Engineering*

Yijun Liu

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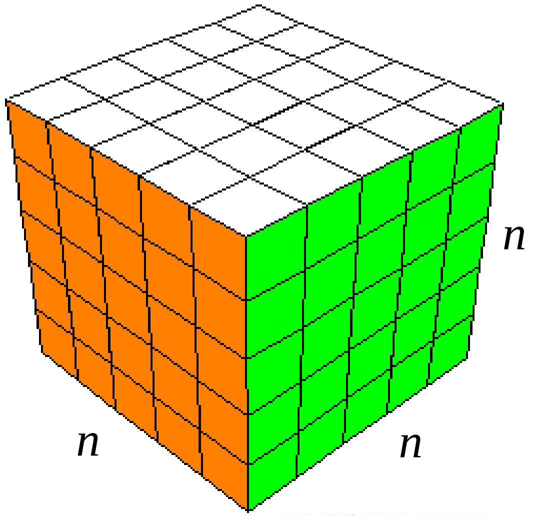
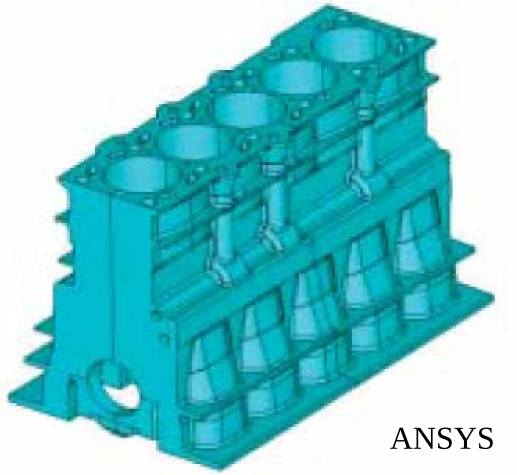
Available at: http://www.yijunliu.com/Research/BEM_Introduction.pdf

The Boundary Element Method (BEM)

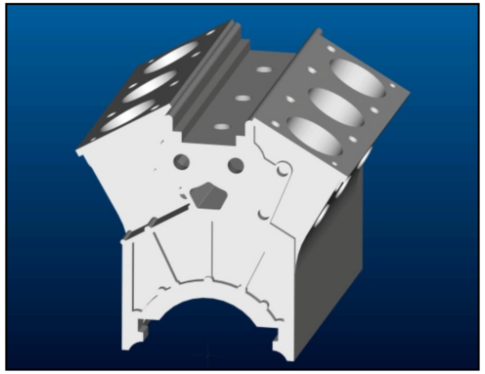
- ***Boundary element method*** applies surface elements on the boundary of a 3-D domain and line elements on the boundary of a 2-D domain. The number of elements is $O(n^2)$ as compared to $O(n^3)$ in other domain based methods (n = number of elements needed per dimension).
- BEM is good for problems with complicated geometries, stress concentration problems, infinite domain problems, wave propagation problems, and many others.
- ***Finite element method*** can solve a model with 1 million DOFs on a PC with 1 GB RAM.
- ***Fast multipole BEM*** can also solve a model with 1 million DOFs on a PC with 1 GB RAM. However, these DOFs are on the *boundary* of the model only, which would require 1 *billion* DOFs for the corresponding domain model.



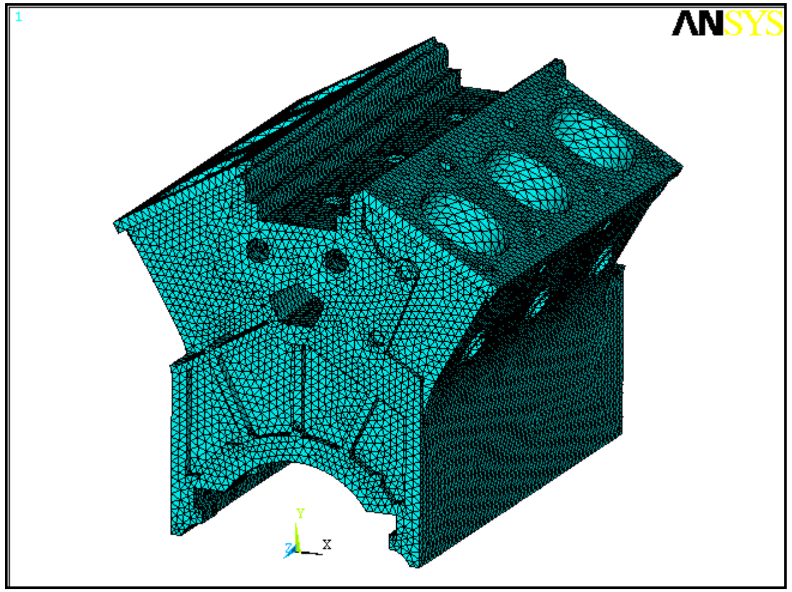


ANSYS

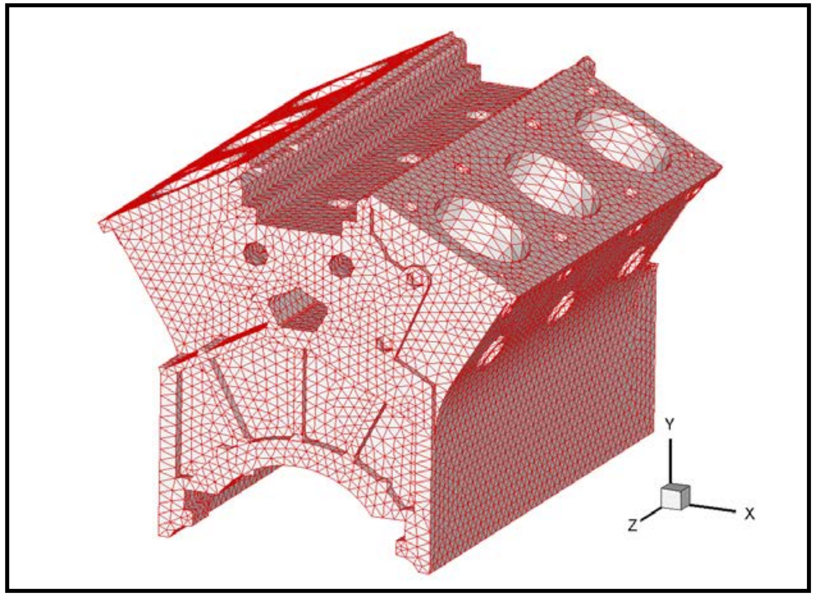
A Comparison of the FEM and BEM
- An Engine Block Model



- Heat conduction of a V6 engine model is studied.
- ANSYS is used in the FEM study.
- Fast multipole BEM is used in the BEM study.
- A linear temperature distribution is applied on the six cylindrical surfaces



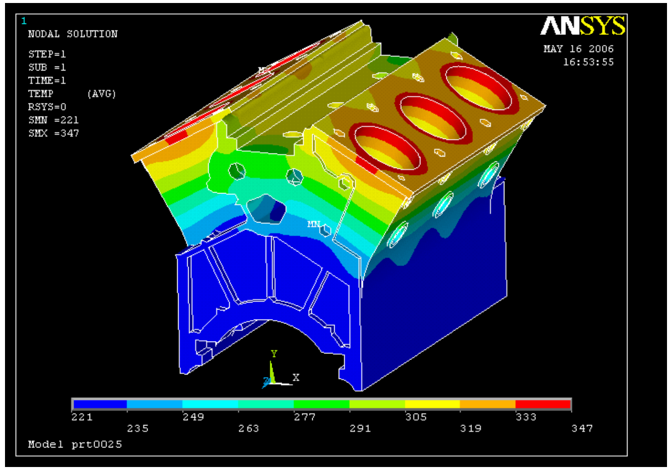
FEM (363,180 volume elements)

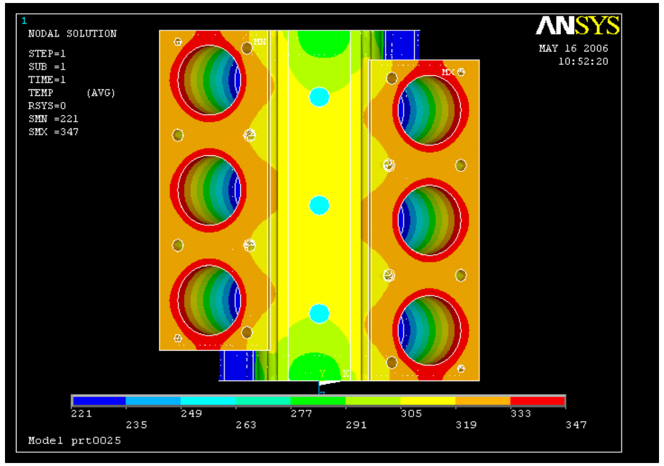


BEM (42,169 surface elements)

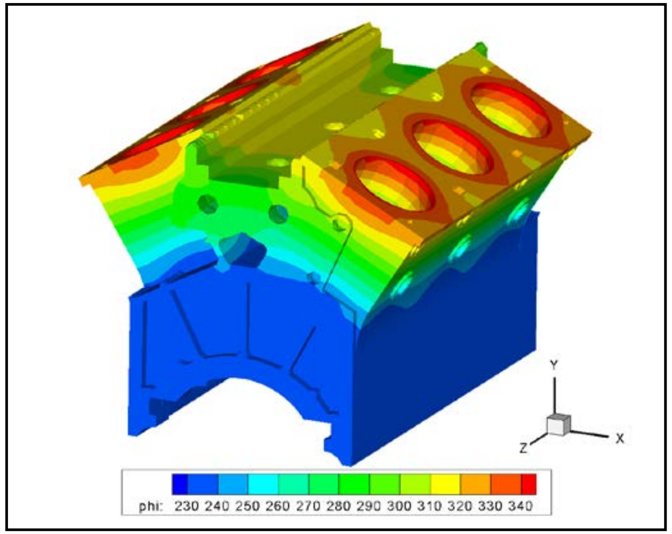
A Comparison of the FEM and BEM with An Engine Block Model (Cont.)

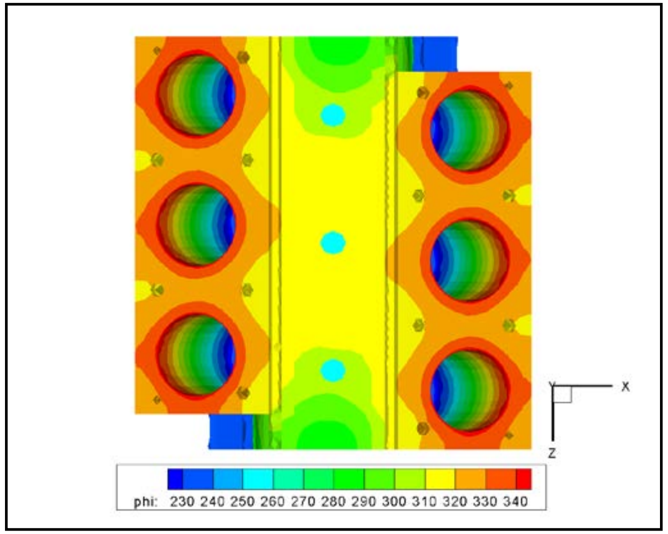
FEM Results
(50 min.)



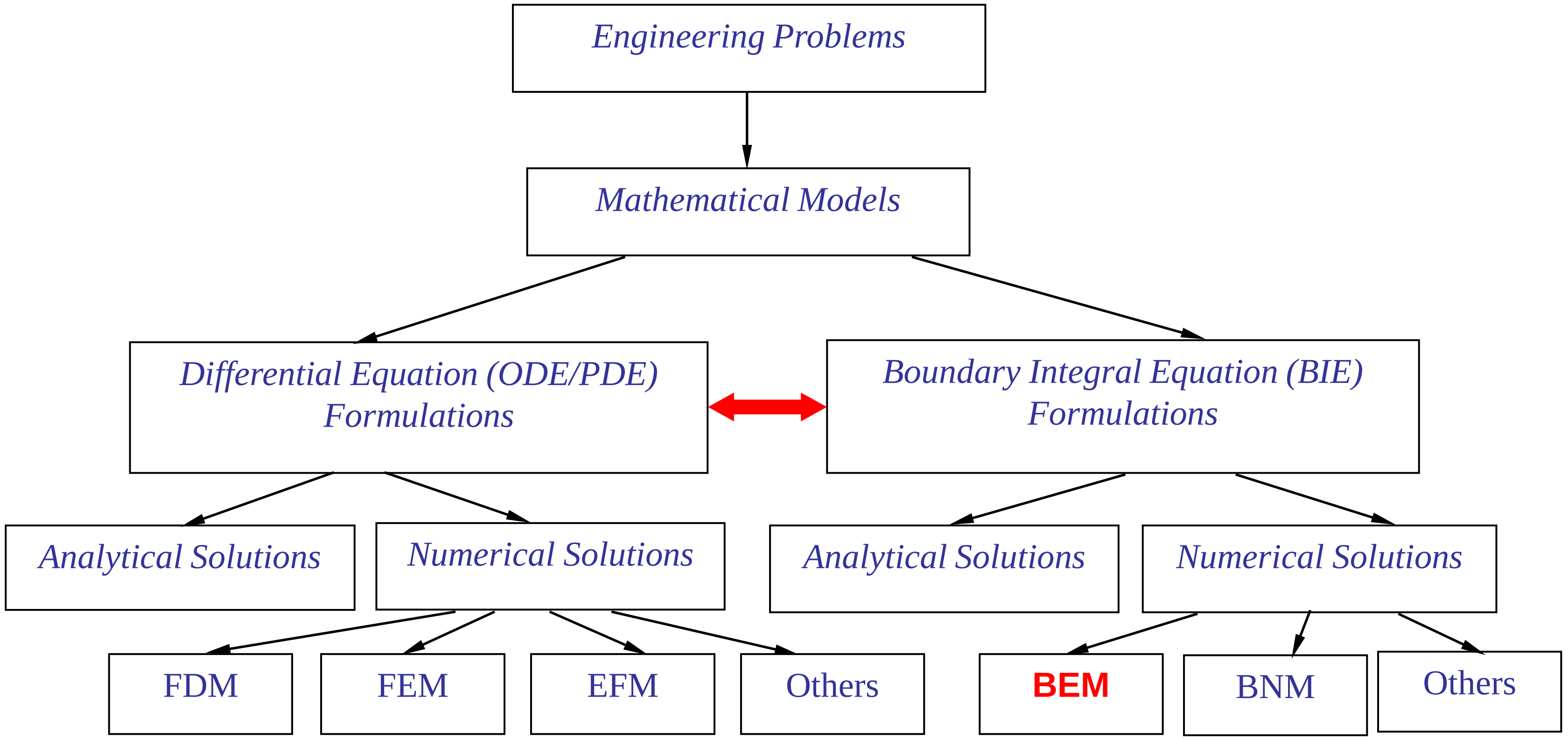


BEM Results
(16 min.)

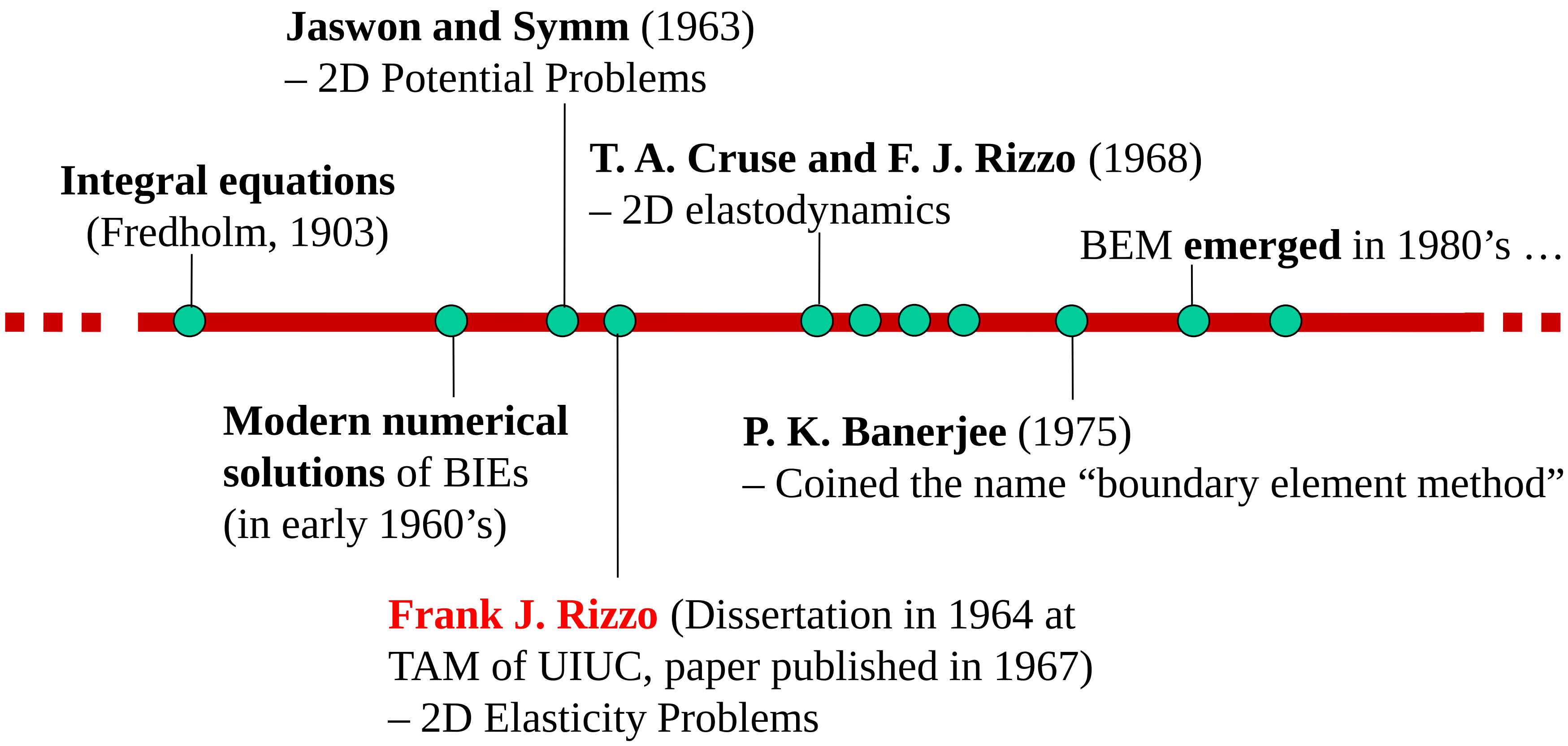




Two Different Routes in Computational Mechanics



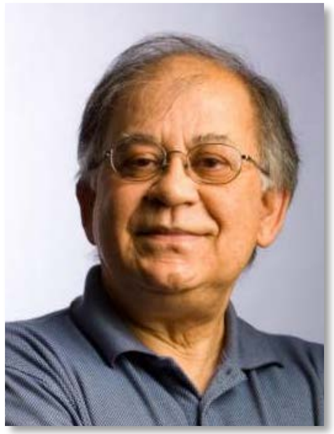
A Brief History of the BEM



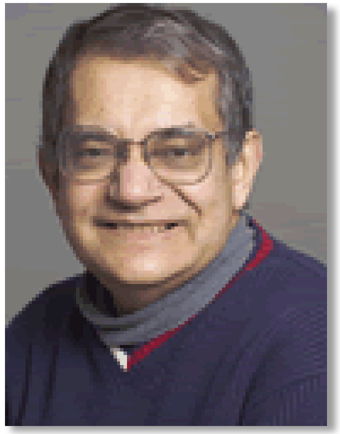
Pioneers in the BIE/BEM Research in the US

Frank J. Rizzo
U Washington
U Kentucky
Iowa State U
U Illinois

Thomas A. Cruse
Boeing
CMU
Pratt & Whitney
SwRI
Vanderbilt U
AFRL



P. K. Banerjee
U Wales, UK
SUNY - Buffalo



Subrata Mukherjee
Cornell U

… Others

Pioneers in Early BIE/BEM Research in China

… Others

杜庆华

清华大学

姚振汉

清华大学

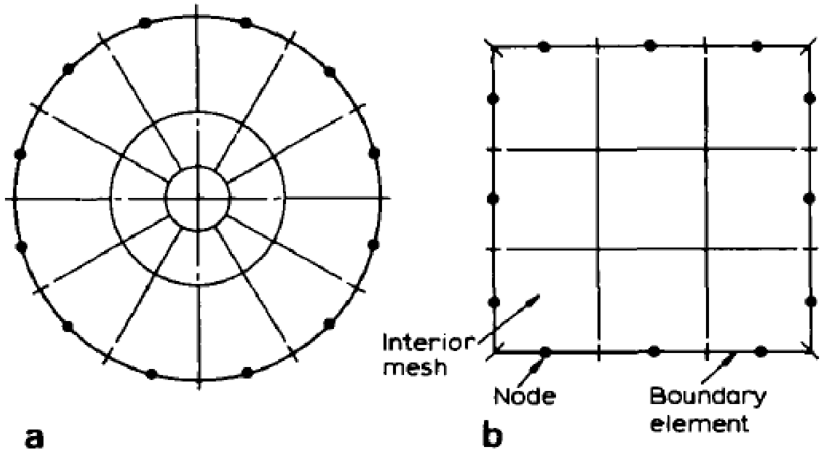
叶天麒

西北工业大学

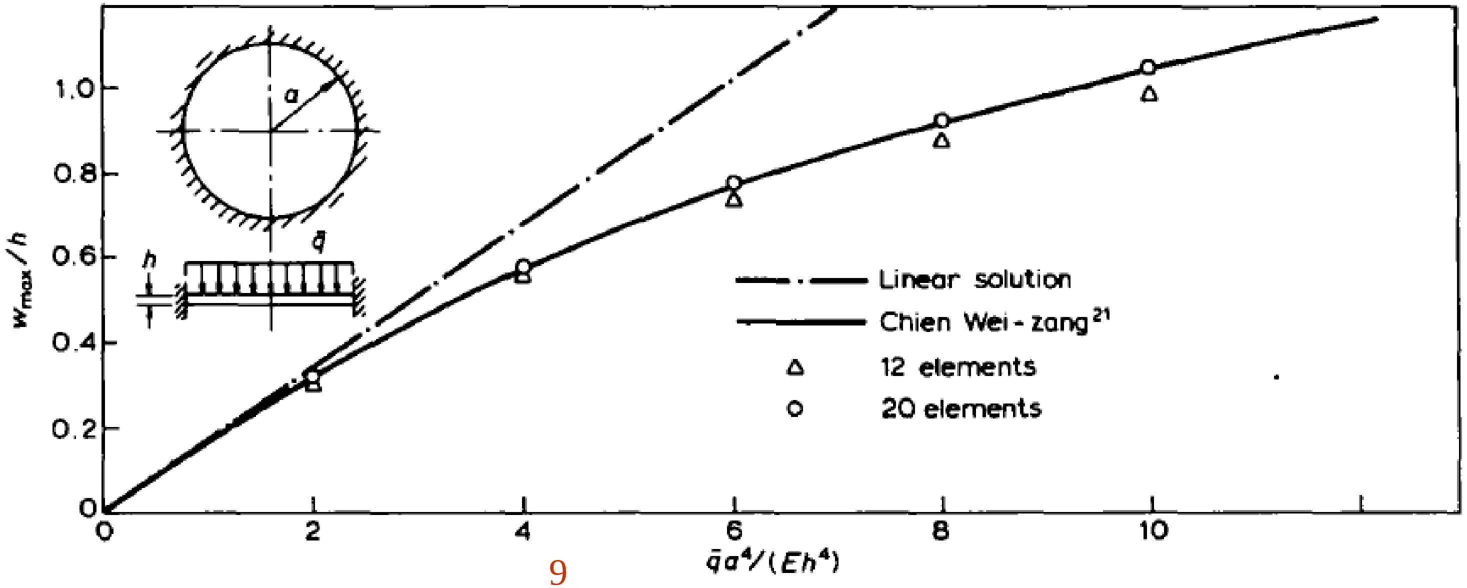
祝家麟

重庆大学

My Earlier BEM Research
– Analysis of Large-Deflection of Elastic Plates



MS thesis research at Northwestern Polytechnical University (NPU), Xi’an, China.
Paper published in: *Applied Mathematical Modelling*, **9**, 183-188 (1985).

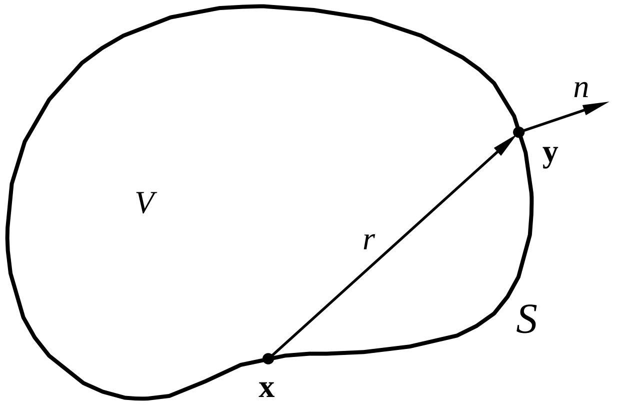


Formulation: The Potential Problem

- Governing Equation

$$\nabla^2 u(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in V;$$

with given boundary conditions on *S*



- The Green's function for potential problem

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right), \quad \text{in 2D;}$$

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi r}, \quad \text{in 3D.}$$

- Boundary integral equation formulation

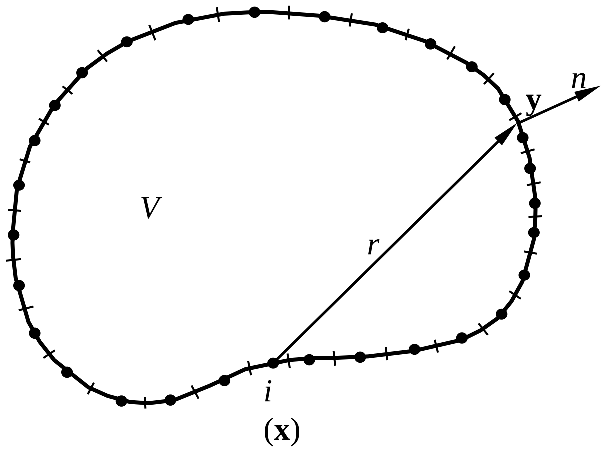
$$C(\mathbf{x})u(\mathbf{x}) = \int_S \left[G(\mathbf{x},\mathbf{y})q(\mathbf{y}) - F(\mathbf{x},\mathbf{y})u(\mathbf{y})\right]dS(\mathbf{y}), \qquad \forall \mathbf{x} \in V \text{ or } S,$$

where $q = \partial u/\partial n, \; F = \partial G/\partial n.$

- Comments: The BIE is exact due to the use of the Green's function;
  Note the singularity of the Green's function $G(\mathbf{x},\mathbf{y})$.

Formulation: The Potential Problem (Cont.)

- Discretize boundary *S* using *N* boundary elements:
  - line elements for 2D problems;
  - surface elements for 3D problems.



- The BIE yields the following BEM equation

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1N} \\ f_{21} & f_{22} & \cdots & f_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N1} & f_{N2} & \cdots & f_{NN} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{Bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1N} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NN} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{Bmatrix}$$

- Apply the boundary conditions to obtain

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix}, \quad \text{or} \quad \mathbf{Ax} = \mathbf{b}$$

Each node/element interacts with all other node/element directly.

The number of operations is of order *O*(N^2).

Storage is also of order *O*(N^2).

Advantages and Disadvantages of the BEM

Advantages:

- **Accuracy** – due to the semi-analytical nature and use of integrals in the BIEs
- **More efficient** in meshing due to the reduction of dimensions
- Good for **stress concentration** and **infinite domain** problems
- Good for modeling **thin shell-like structures**/models of materials
- **Neat** … (integration, superposition, boundary solutions for BVPs)

Disadvantages:

- Conventional BEM matrices are *dense and nonsymmetrical*
- Solution time is *long* and memory size is *large* (Both are $O(N^2)$)
- Used to be limited to for solving small-scale BEM models (Not anymore!)

The Solution:

- Various fast solution methods to improve the computational efficiencies of the BEM

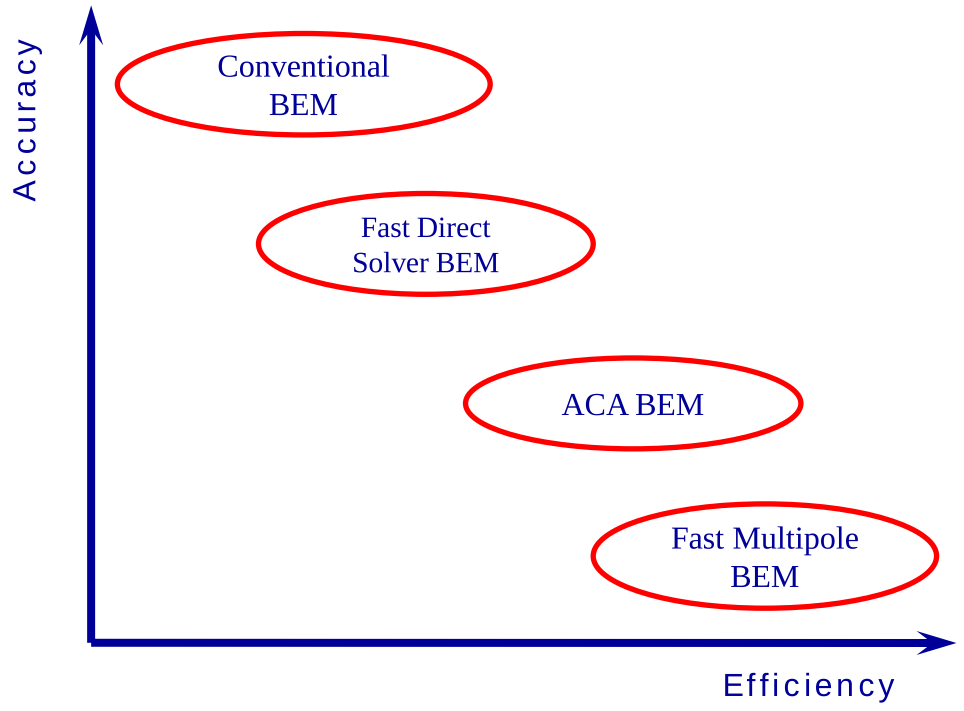
Overview of the Fast BEM

Methods:

- *Fast multipole method* (FMM) (Rokhlin and Greengard, 1980s; Nishimura, 2001)
- *Adaptive cross approximation* (ACA) method (Bebendorf, *et al*., 2000)
- *Fast direct solvers* (Martinsson, Rokhlin, Greengard, Darve, *et al*.)

Techniques:

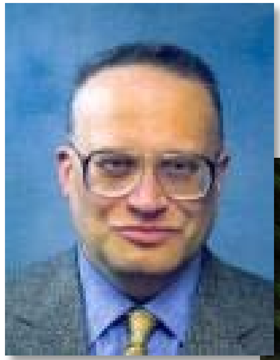
- *Domain decomposition* (new multidomain BEM, Liu & Huang, 2016)
- *Parallel computing* on CPU or GPU



Reference: Y. J. Liu, S. Mukherjee, N. Nishimura, M. Schanz, W. Ye, A. Sutradhar, E. Pan, N. A. Dumont, A. Frangi and A. Saez, "Recent advances and emerging applications of the boundary element method," ASME *Applied Mechanics Review*, **64**, No. 5 (May), 1–38 (2011).

Fast Multipole Method (FMM)

- FMM can reduce the cost (CPU time & storage) for BEM to $O(N)$
- Pioneered by **Rokhlin** and **Greengard** (mid of 1980's)
- Ranked among the top ten algorithms of the 20th century (with FFT, QR, …) in computing
- **Greengard**'s book: *The Rapid Evaluation of Potential Fields in Particle Systems*, MIT Press, 1988
- An earlier review by **Nishimura**: ASME *Applied Mechanics Review*, July 2002
- A newer review by **Liu, Mukherjee, Nishimura, Schanz, Ye**, et al, ASME *Applied Mechanics Review*, May 2011
- A book by **Liu**: *Fast Multipole Boundary Element Method – Theory and Applications in Engineering*, Cambridge University Press, 2009



Rokhlin

Greengard

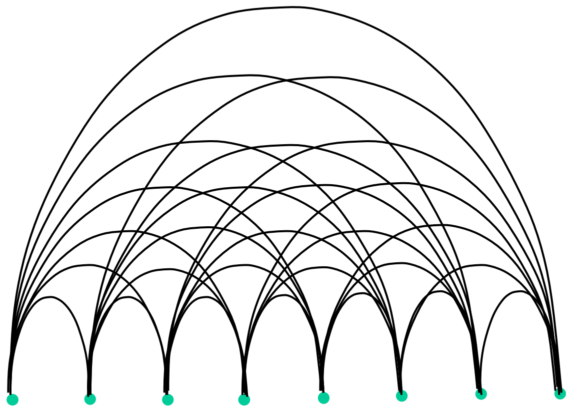
Chew

Nishimura

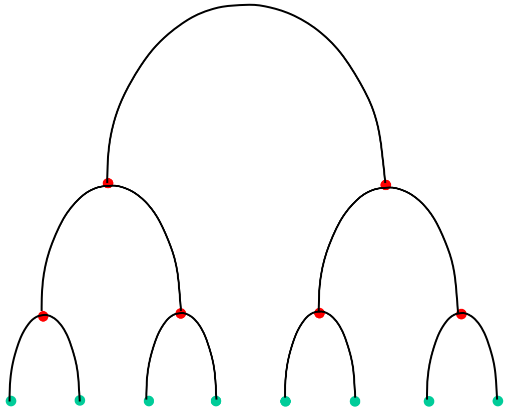
Fast Multipole Method (FMM): The Simple Idea

Apply iterative solver (GMRES) and accelerate matrix-vector multiplications by replacing element-element interactions with cell-cell interactions.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix}, \quad \text{or} \quad \mathbf{Ax} = \mathbf{b}$$



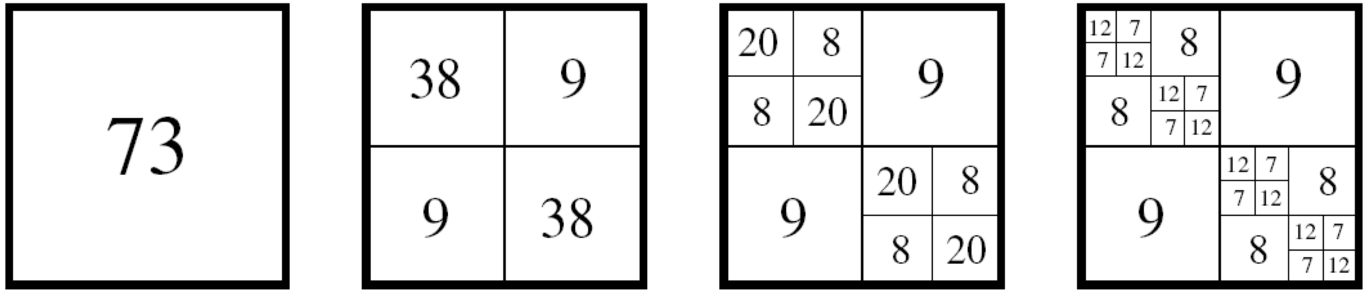
Conventional BEM approach ($O(N^2)$)



FMM BEM approach ($O(N)$ for large N)

Adaptive Cross Approximation (ACA)

- Hierarchical decomposition of a BEM matrix:



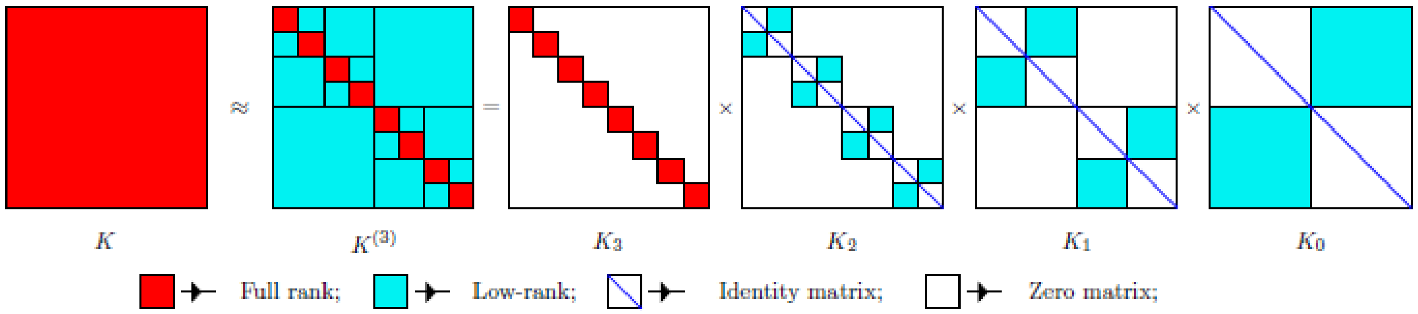
(from Rjasanow and Steinbach, 2007)

- A lower-rank submatrix **A** away from the main diagonal can be represented by a few selected columns (**u**) and rows ($\mathbf{v}^T$) (crosses) based on error estimates:

$$\mathbf{A}_k \approx \sum_{\alpha=1}^{k} \frac{1}{\gamma_\alpha} \mathbf{u}_\alpha \mathbf{v}_\alpha{}^T, \quad \text{with } \gamma = \mathbf{A}(i, j),\ \ \mathbf{u} = \mathbf{A}(:, j),\ \ \mathbf{v} = \mathbf{A}(i,:)$$

- The process is independent of the kernels (or 2-D/3-D)
- Can be integrated with iterative solvers (GMRES)

Fast Direct Solver



$$\mathbf{K} \approx \mathbf{K}^{l_m} = \mathbf{K}_{l_m}\mathbf{K}_{l_m-1}\cdots\mathbf{K}_0, \qquad \mathbf{Kx}=\mathbf{b} \quad \Rightarrow \quad \mathbf{x} = \mathbf{K}_0^{-1}\cdots\mathbf{K}_{l_m-1}^{-1}\mathbf{K}_{l_m}^{-1}\mathbf{b}$$

Sherman-Morrison-Woodbury formula:

$$\left(\mathbf{I}+\mathbf{U}\mathbf{V}^T\right)^{-1} = \mathbf{I} - \mathbf{U}\left(\mathbf{I}+\mathbf{V}^T\mathbf{U}\right)^{-1}\mathbf{V}^T$$

which is efficient if .

$$p \ll N \text{ for } \mathbf{U} \text{ and } \mathbf{V} \in \mathbb{R}^{N\times p}$$

See reference for application to the BEM matrices.

$$\mathbf{K}_i^l = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{U}}_{2i-1}^{l+1}\left(\mathbf{V}_{2i-1}^{l+1}\right)^T \\ \tilde{\mathbf{U}}_{2i}^{l+1}\left(\mathbf{V}_{2i}^{l+1}\right)^T & \mathbf{I} \end{bmatrix}$$

$$= \mathbf{I} + \begin{bmatrix} \tilde{\mathbf{U}}_{2i-1}^{l+1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}_{2i}^{l+1} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \left(\mathbf{V}_{2i-1}^{l+1}\right)^T \\ \left(\mathbf{V}_{2i}^{l+1}\right)^T & \mathbf{0} \end{bmatrix}$$

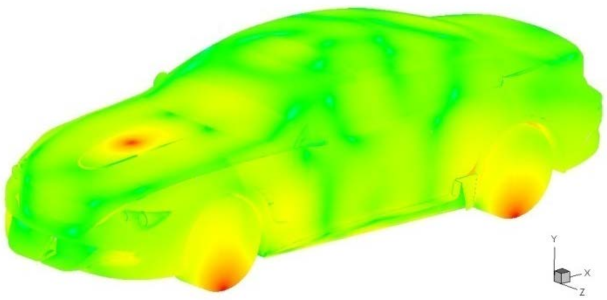
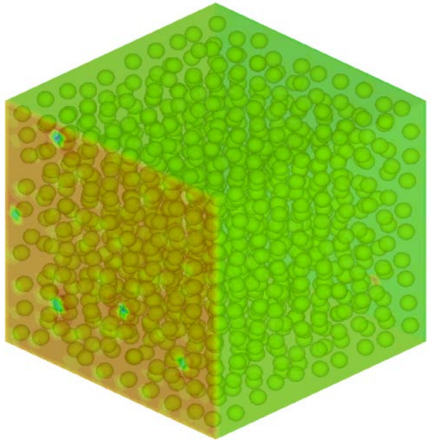
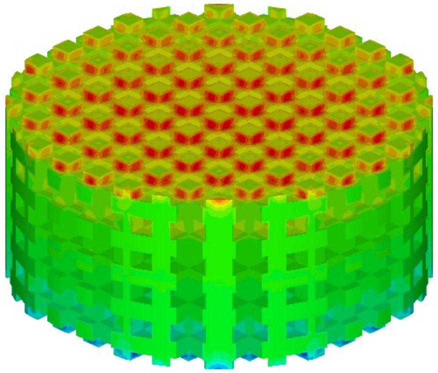
$$= \mathbf{I} + \mathbf{U}_i^l\left(\mathbf{V}_i^l\right)^T$$

Reference: S. Huang and Y. J. Liu, "A new fast direct solver for the boundary element method," *Computational Mechanics*, **60**, No. 3, 379–392 (2017).

Some Applications of the Fast Multipole Boundary Element Method

- 2-D/3-D potential problems.
- 2-D/3-D elasticity problems.
- 2-D/3-D Stokes flow problems.
- 2-D/3-D acoustics problems.
- Applications in modeling porous materials, fiber-reinforced composites and micro-electro-mechanical systems (MEMS).

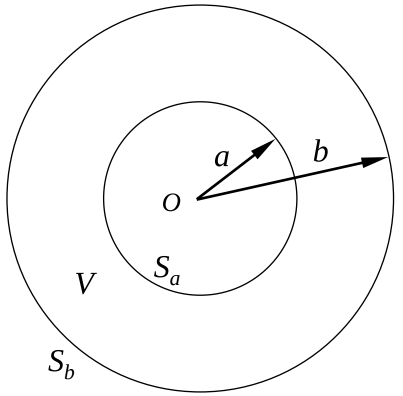
- All software packages used here can be downloaded from **www.yijunliu.com**.

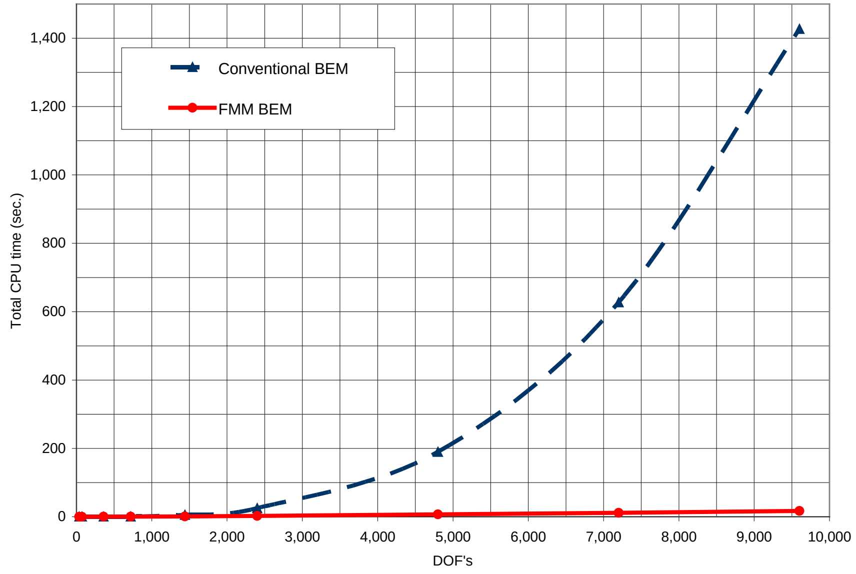




2-D Potential: Accuracy and Efficiency of the Fast Multipole BEM

Results for a simple potential problem in an annular region V

N	q_a	
	FMM BEM	*Conventional BEM*
36	-401.771619	-401.771546
72	-400.400634	-400.400662
360	-400.014881	-400.014803
720	-400.003468	-400.003629
1440	-400.000695	-400.000533
2400	-400.001929	-400.000612
4800	-400.001557	-400.000561
7200	-399.997329	-399.998183
9600	-399.997657	-399.996874
Analytical Solution	-400.0	





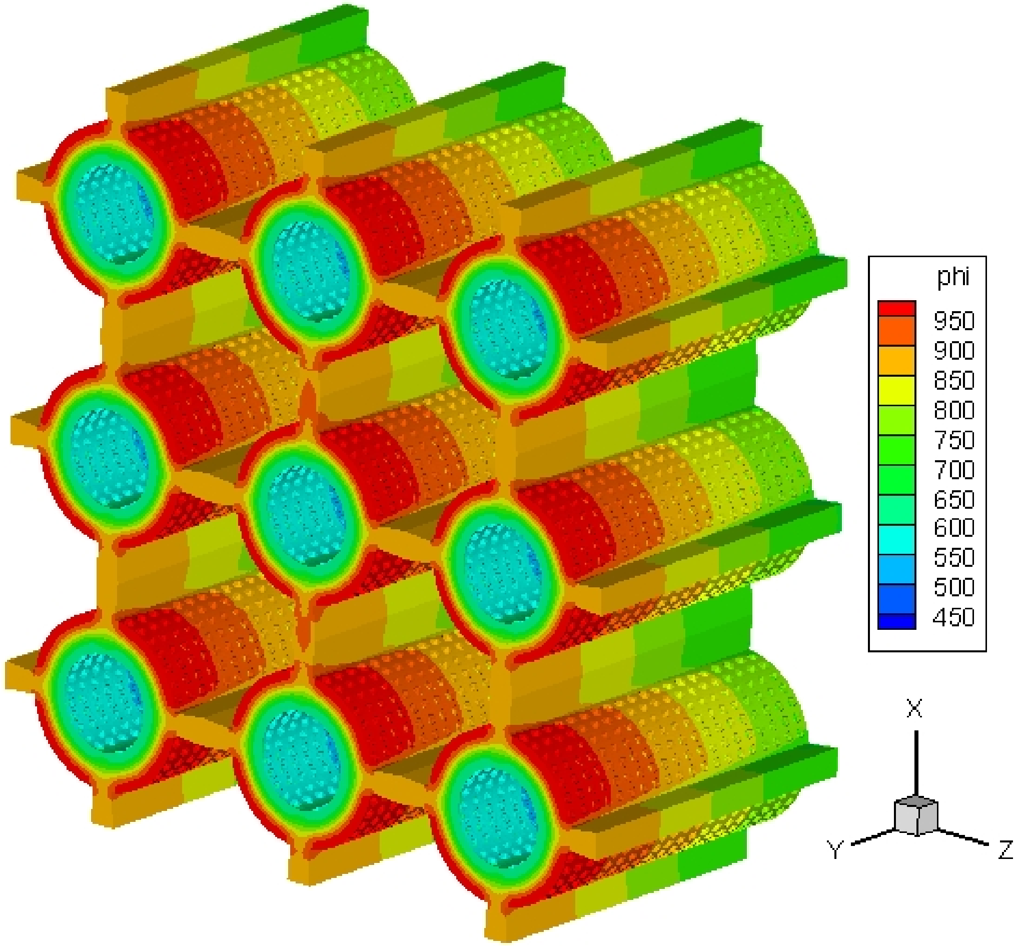
3-D Potential: Modeling of Fuel Cells

Thermal Analysis of Fuel Cell (SOFC) Stacks

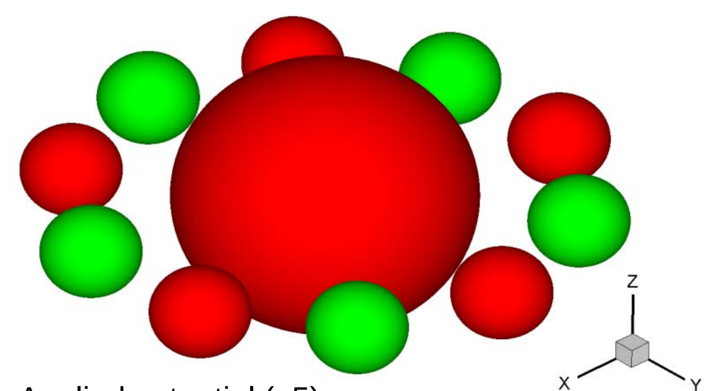
There are 9,000 small side holes in this model

Total DOFs = 530,230, solved on a desktop PC with 1 GB RAM)

ANSYS can only model one cell on the same PC

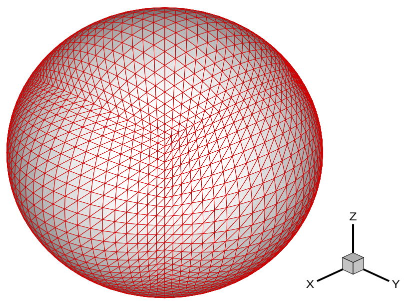


3-D Electrostatic Analysis

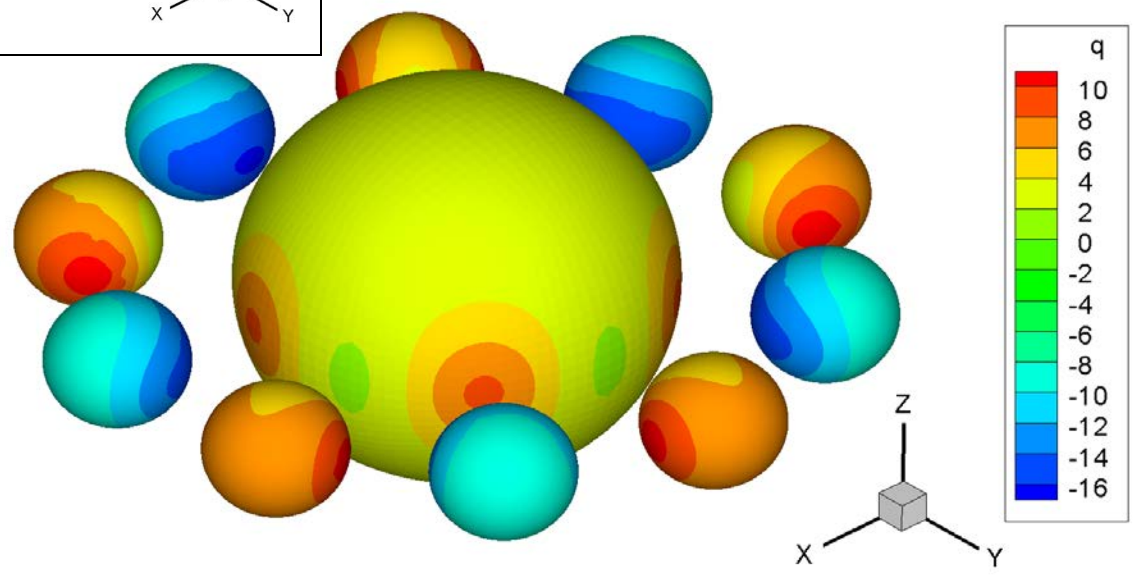


Applied potential (±5)

- **11 conducting spheres.**
- **Forces can be found with the charge density.**
- **Largest model has 118,800 DOFs.**



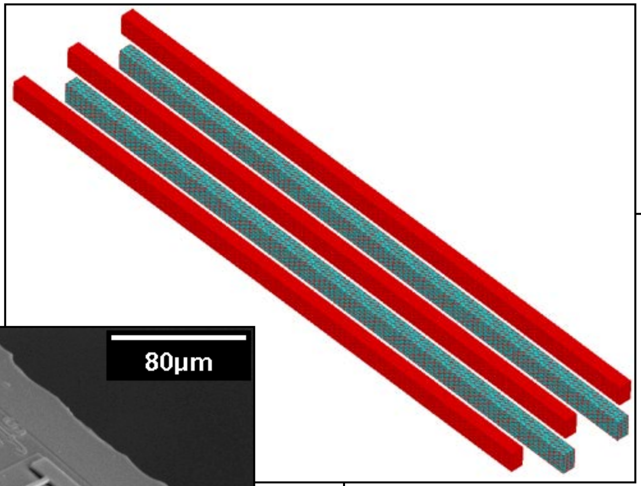
One BEM mesh



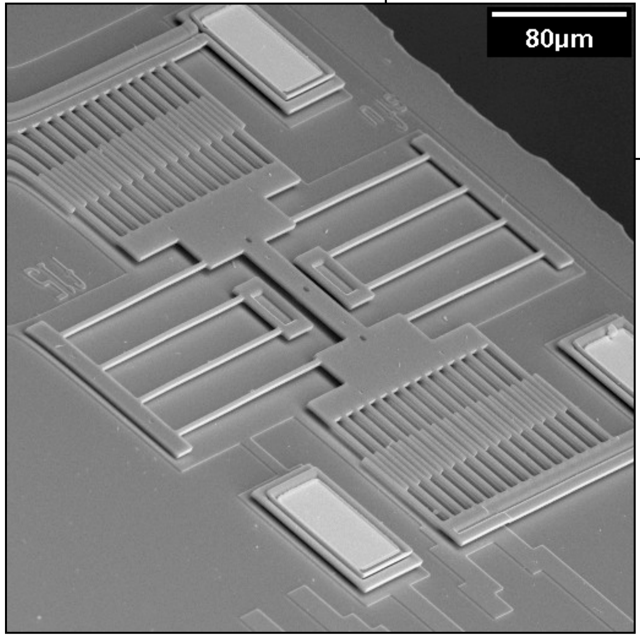
Computed charge density

3-D Electrostatic Analysis (Cont.)

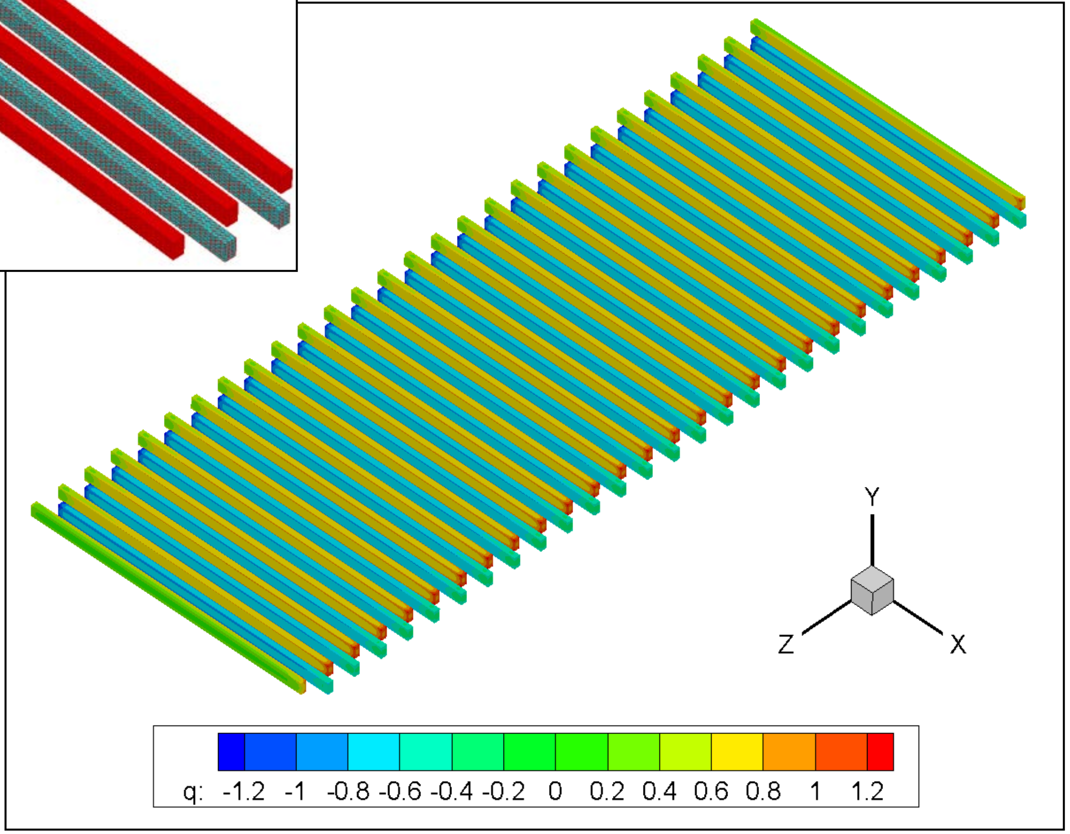
Applications in MEMS



- Beams are applied with +/- voltages.
- Forces can be found with the charge density.
- Model shown has 55 beams (179,300 DOFs).

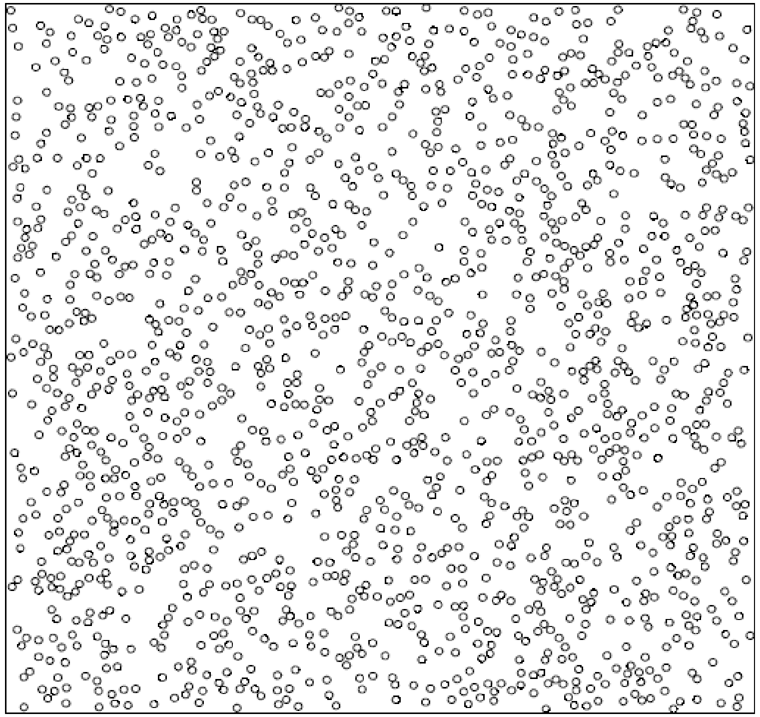


A comb drive



2-D Elasticity: Modeling of Perforated Plates

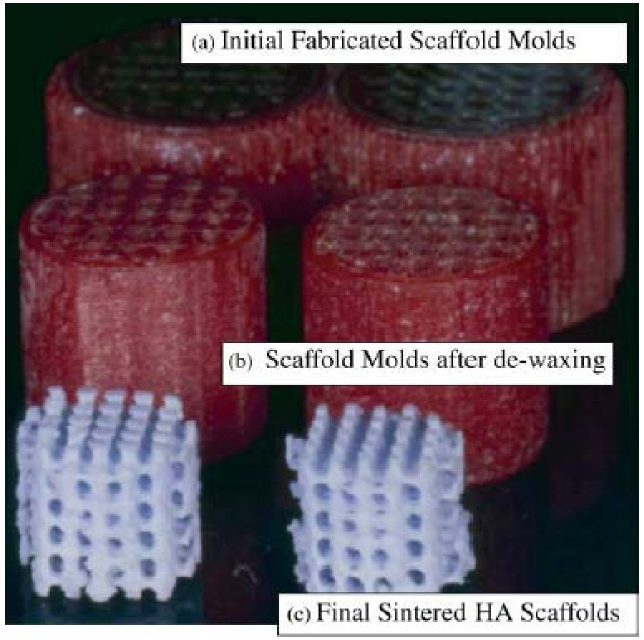
A BEM model of a perforated plate
(with 1,600 holes)



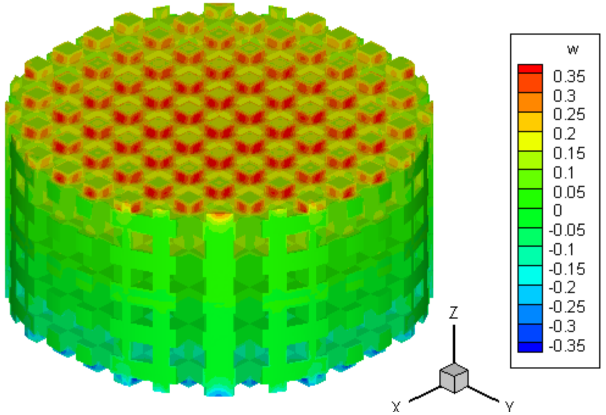
Computed effective Young's modulus
for the perforated plate (x E)

No. Holes	*DOFs*	*Uniformly Distributed Holes*	*Randomly Distributed Holes*
2x2	3,680	0.697892	0.678698
4x4	13,120	0.711998	0.682582
6x6	28,320	0.715846	0.659881
8x8	49,280	0.717643	0.651026
12x12	108,480	0.719345	0.672084
20x20	296,000	0.720634	0.676350
30x30	660,000	0.721255	0.676757
40x40	1,168,000	0.721558	0.675261

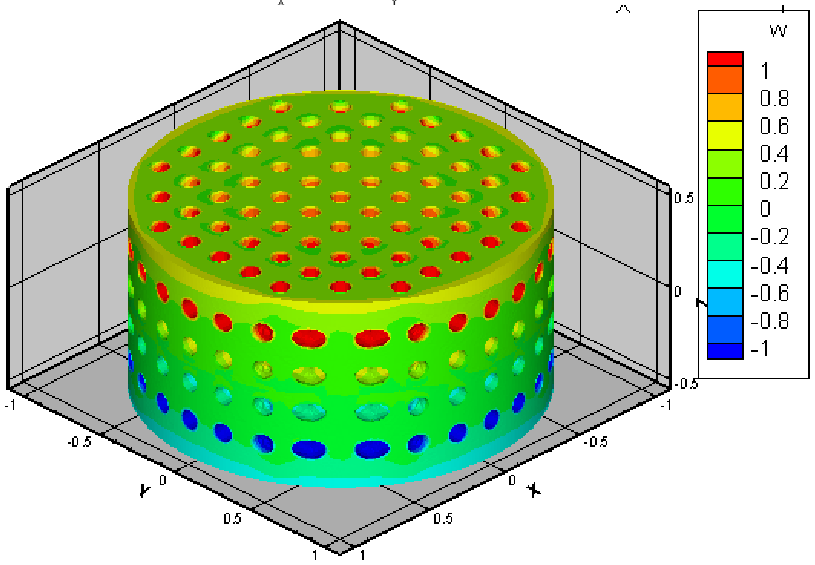
3-D Elasticity: Modeling of Scaffold Materials



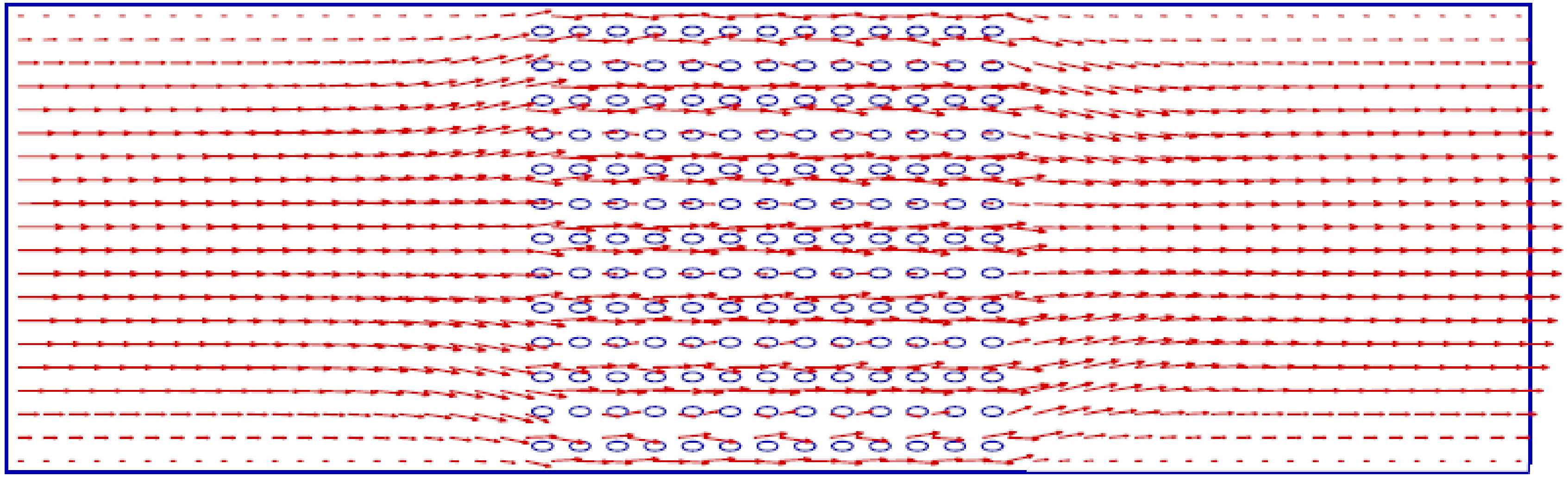
(Hollister, et al, 2002)



Preliminary BEM models and results



2-D Stokes Flow: Multiple Cylinders

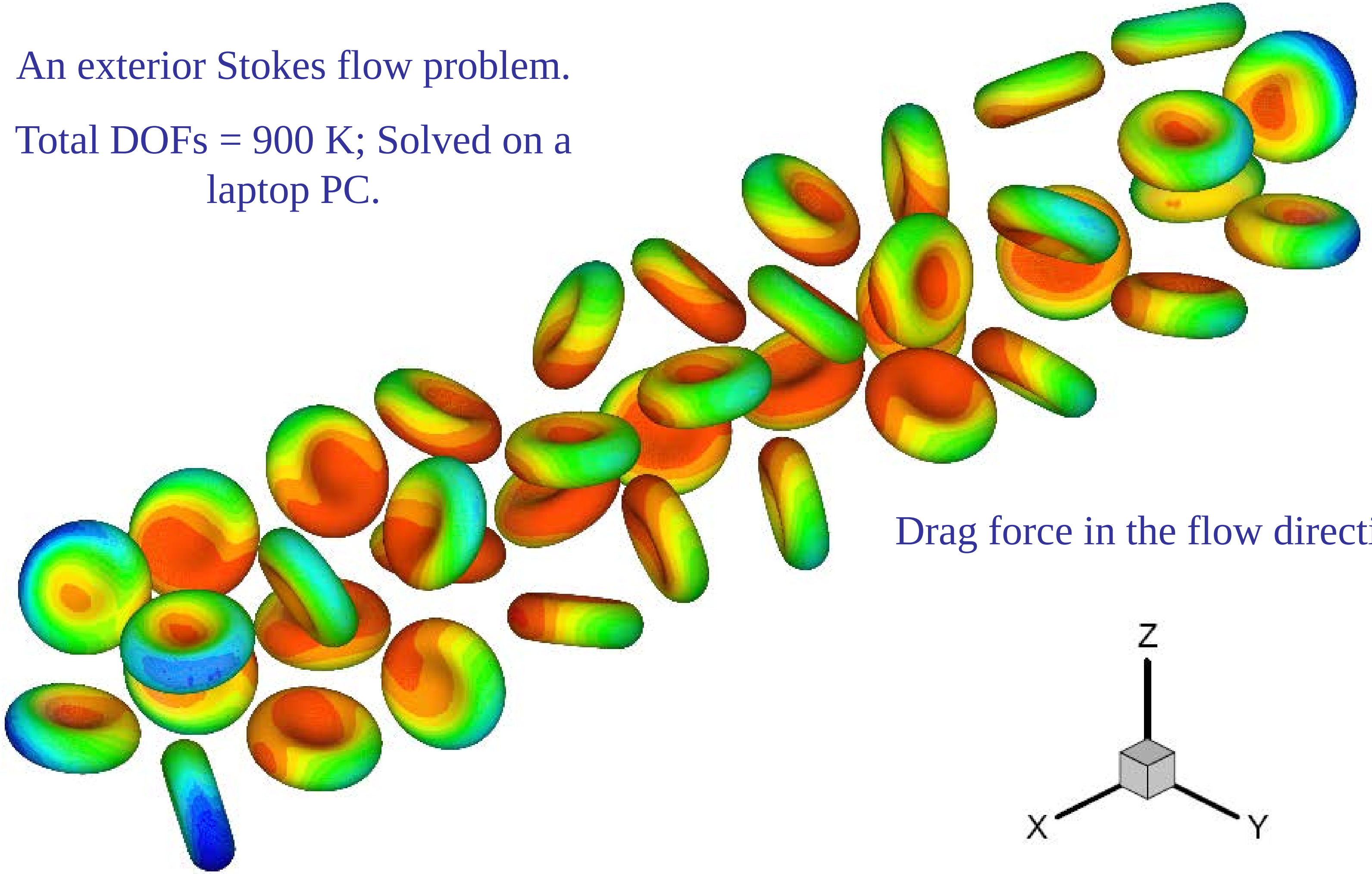


(c) A larger model with 13 x 13 elliptic cylinders and $a = 0.02h$, $b = 0.5a$, DOFs = 103,000.

3-D Stokes Flow: Modeling of RBCs

An exterior Stokes flow problem.

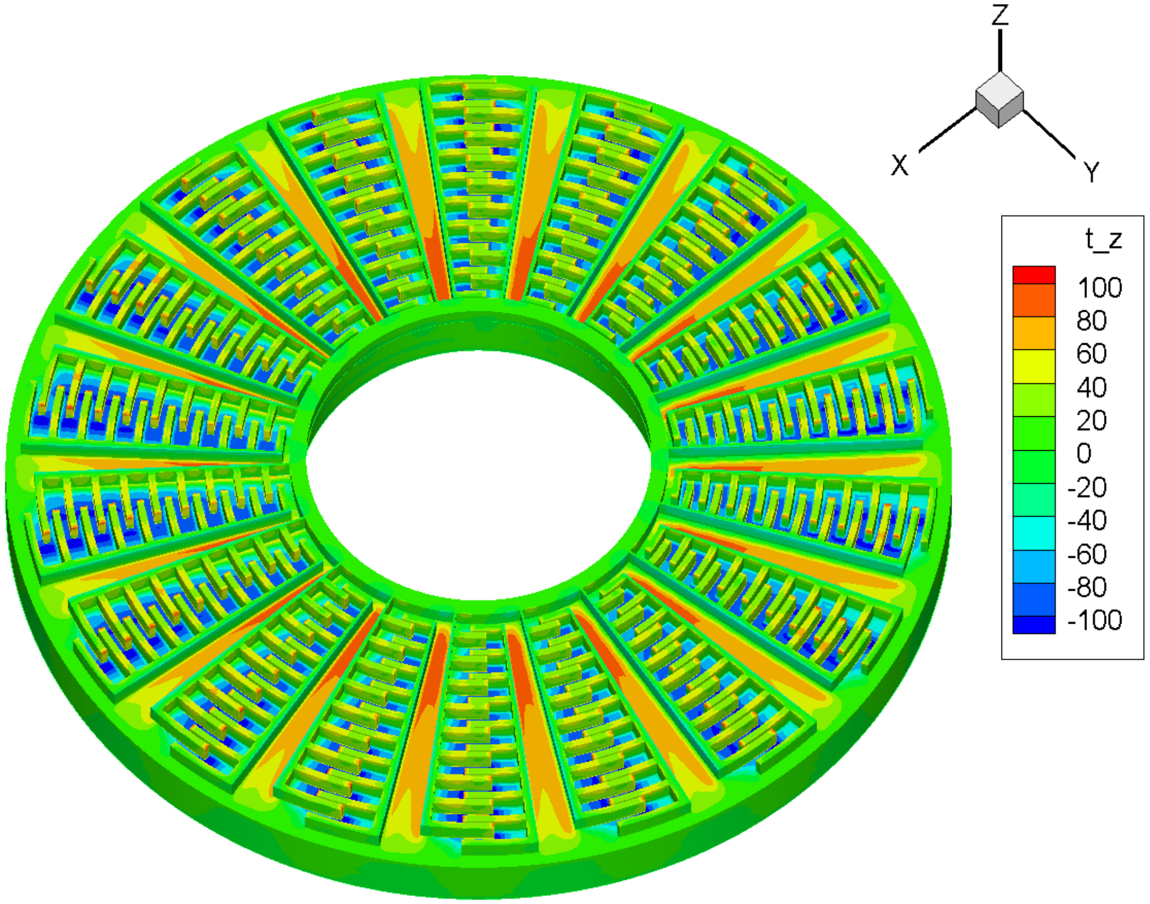
Total DOFs = 900 K; Solved on a laptop PC.



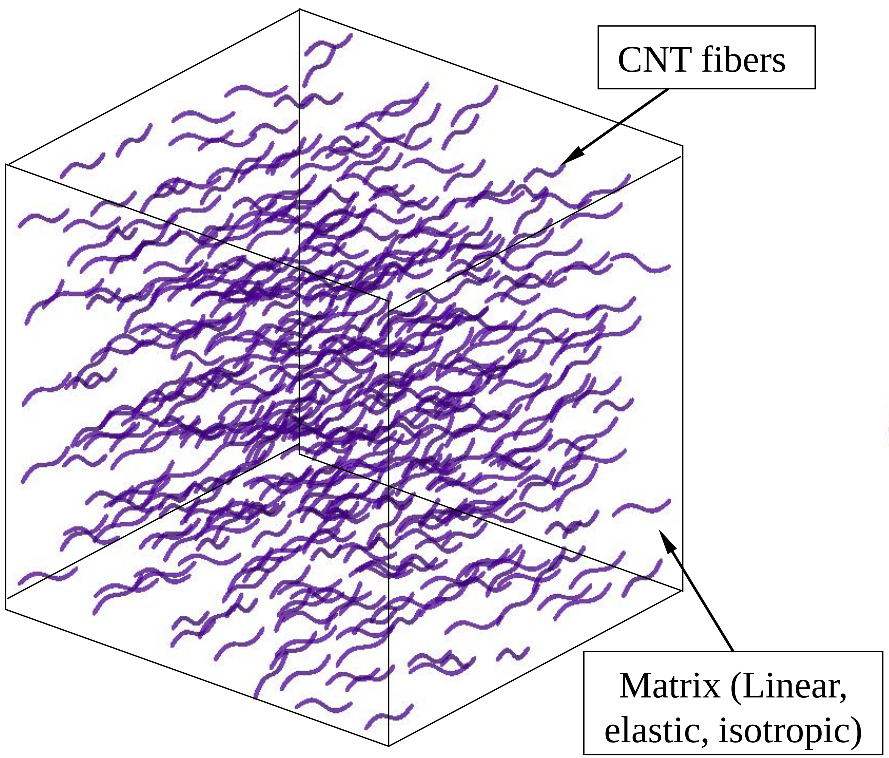
Drag force in the flow direction.

3-D Stokes Flow: MEMS Analysis

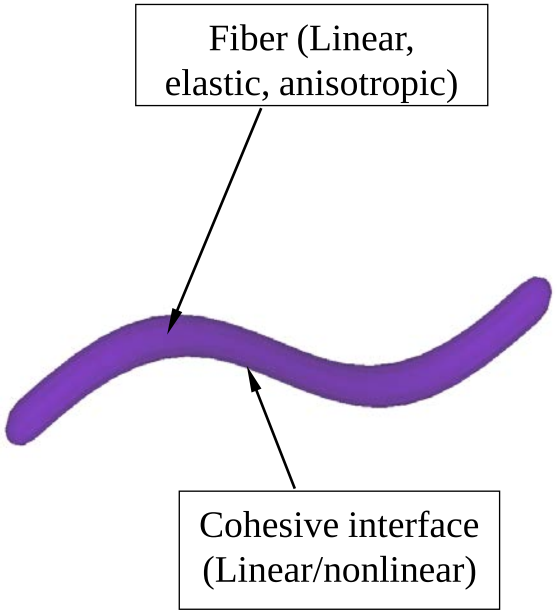
- BEM model with 362,662 elements (1,087,986 total DOFs)
- An angular velocity is applied
- Drag forces are computed
- Solved on a desktop PC



Modeling CNT Composites



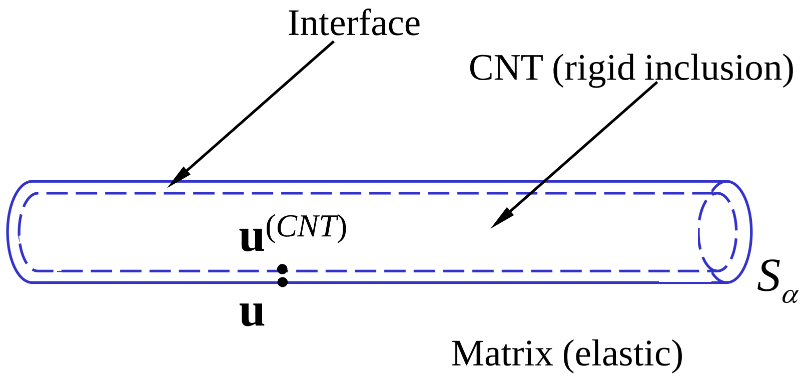
(a) An RVE with many CNT fibers (to be solved by the fast multipole BEM)



(b) Models for the CNTs and interfaces (to be extracted from MD simulations)

A Multiscale Model for CNT Composites

- A rigid-inclusion model is applied to represent the CNT fibers in polymer matrix.
- The cohesive model from MD study is applied for the CNT/polymer interfaces.



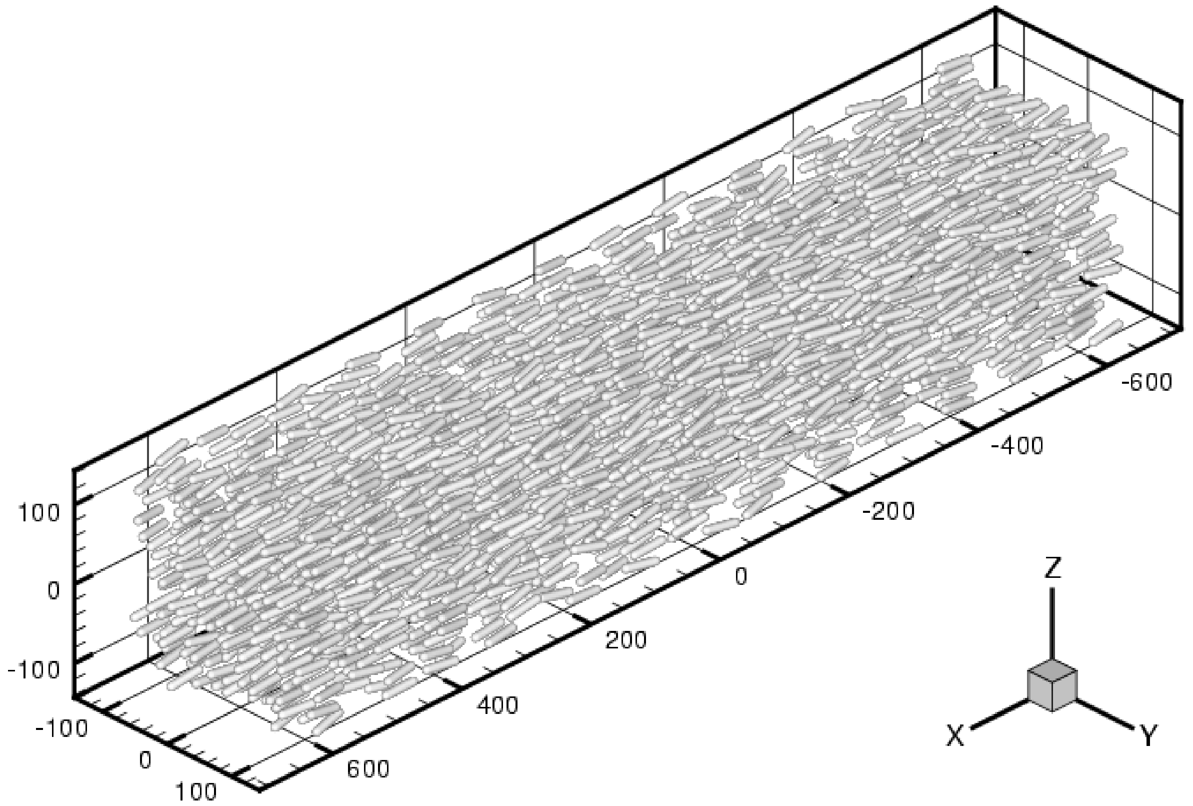
A cohesive interface model:

$$\mathbf{u}^{(CNT)} - \mathbf{u} = \mathbf{C}\mathbf{t}, \quad \forall \mathbf{y} \in S_\alpha,$$

with **C** being the compliance matrix (determined by MD)

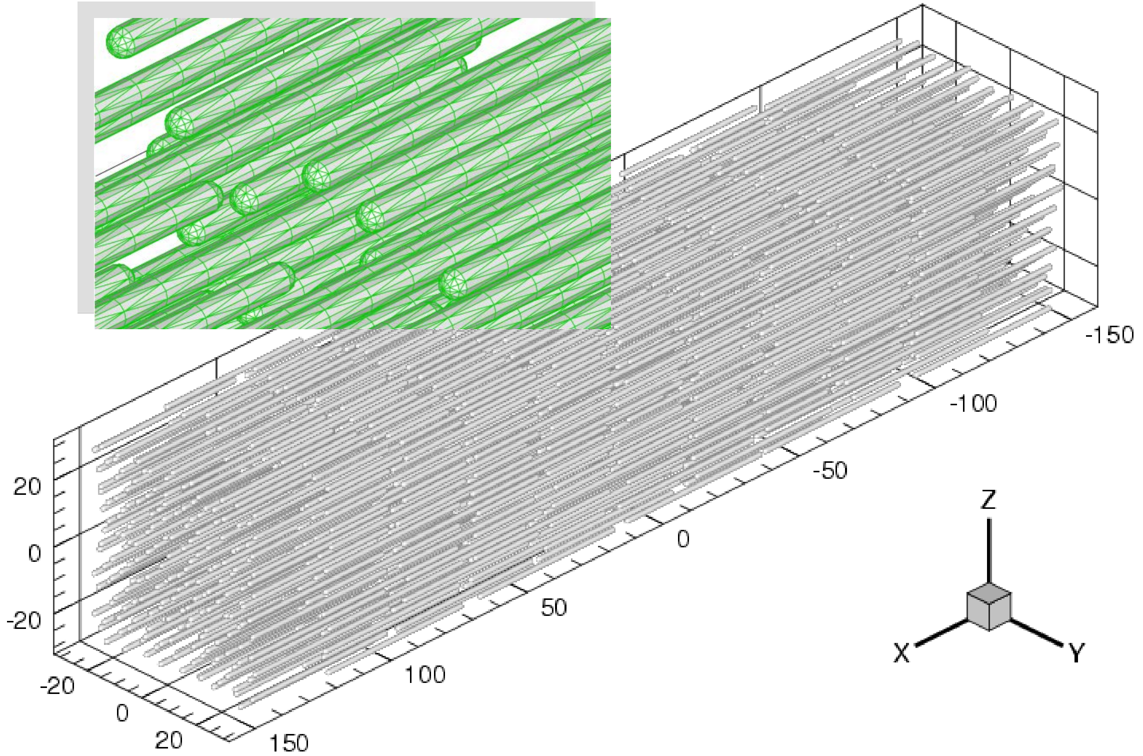
- The fast multipole BEM is applied to solve the large BEM systems.
- This approach is a *first* step toward the more general *multiscale* model with continuum BEM for matrix, and nanoscale MD for CNTs and interfaces.

A Typical RVE Using the BEM



A model containing 2,197 short CNT fibers with the total DOF = 3,018,678

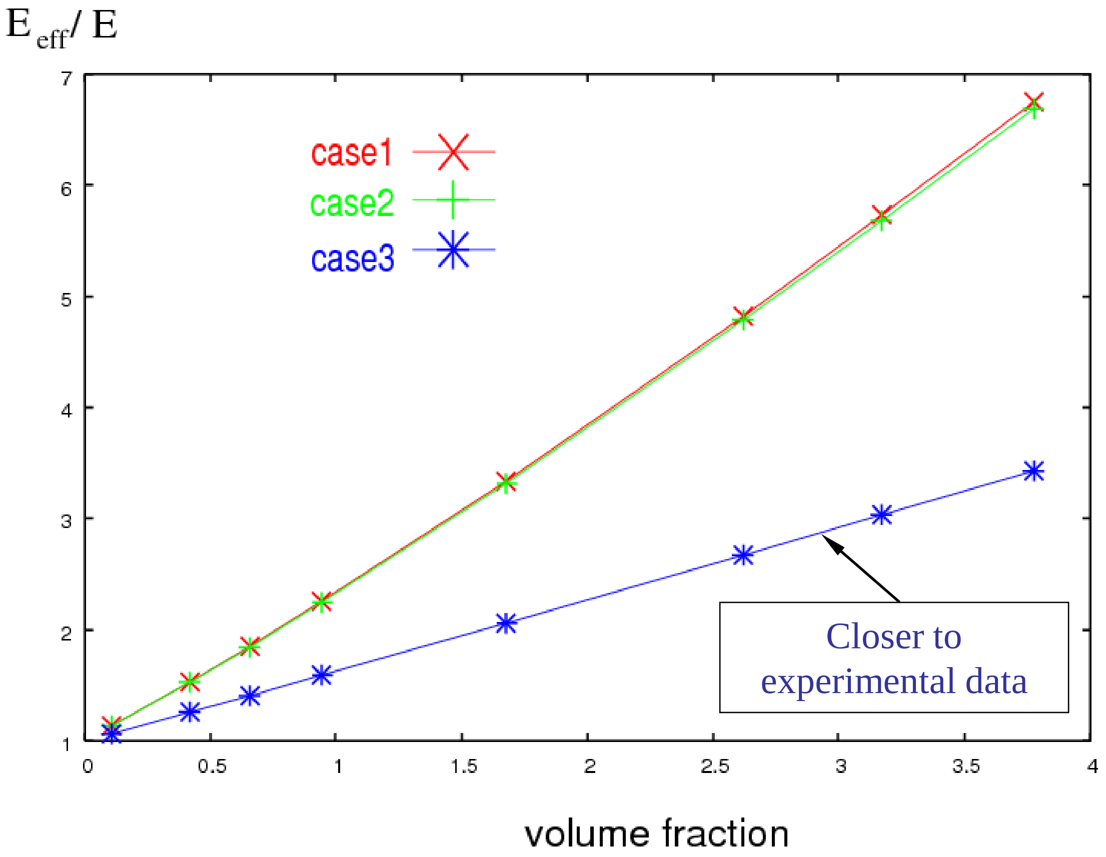
A Very Large BEM Model



An RVE containing 2,000 CNT fibers with the total DOF = 3,612,000 (CNT length = 50 nm, volume fraction = 10.48%). A larger model with 16,000 CNT fibers (8 times of what is shown above) and 28.9M DOFs was solved successfully on a FUJITSU HPC2500 supercomputer at Kyoto University

Modeling of CNT Composites (Cont.)

Effects of the Cohesive Interface



Computed effective moduli of CNT/polymer composites
(same CNT and RVE dimensions as used in the previous perfect bonding case)

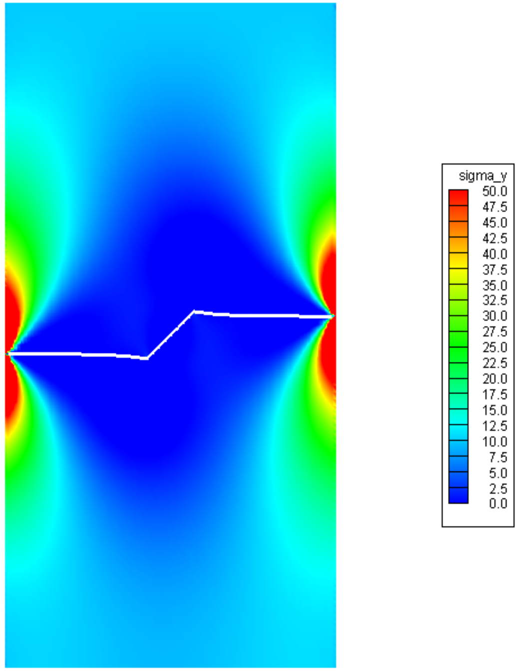
Case 1:
C11=C22=C33=0
(perfect bonding)

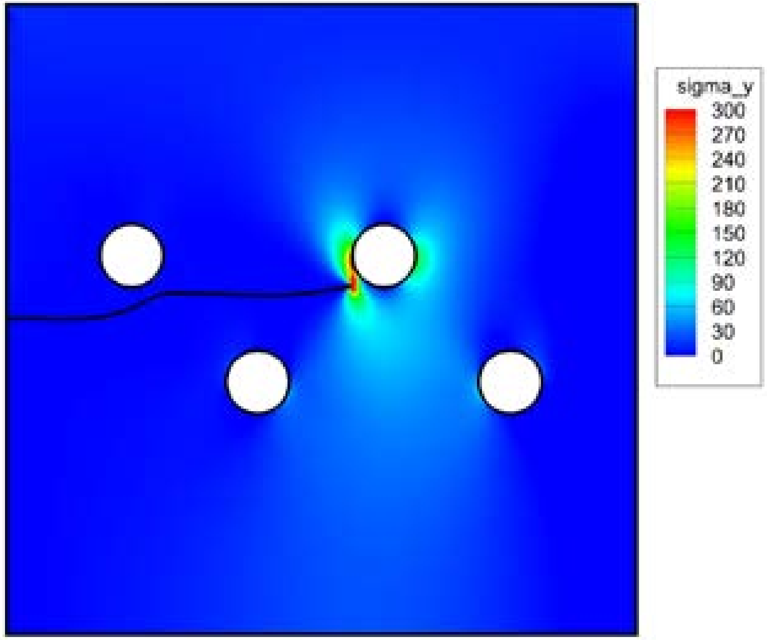
Case 2:
C11=C22=C33=Cr = 0.02157 (large stiffness)

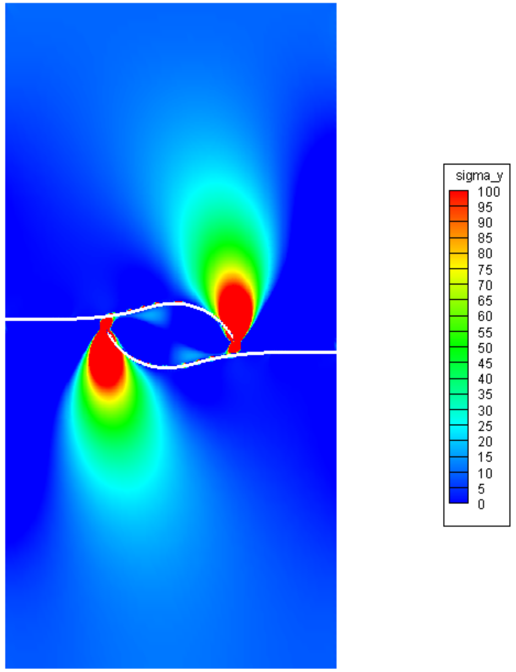
Case 3:
C11=C22=C33=Cz = 3.506 (small stiffness)

Cr, Cz are interface compliance ratios in the radial and longitudinal direction of the fiber, respectively, and are determined from the MD simulations.

BEM in Modeling of Cracks in 2-D/3-D Solids

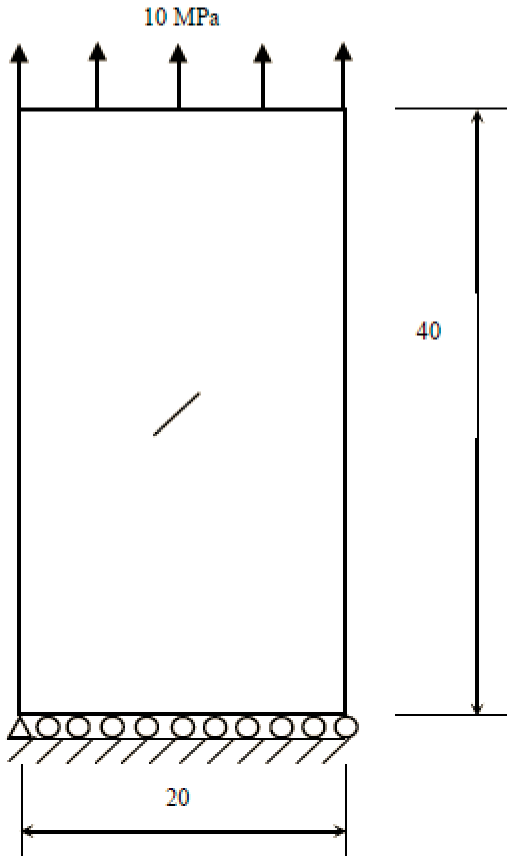


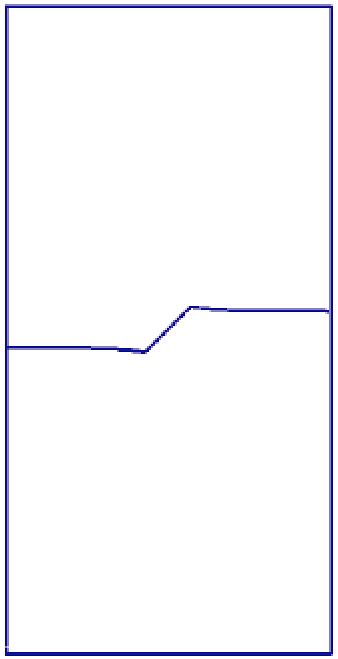




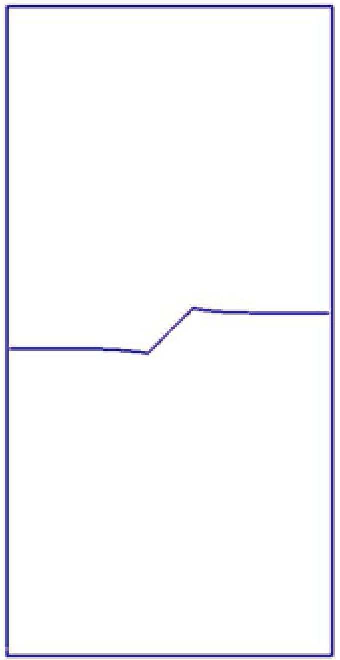
Constant line elements are used (equivalent to the DDM) with analytical integration of all integrals, which is sufficiently accurate and very efficient (just need more elements ☺).

2-D Example: A Benchmark Problem





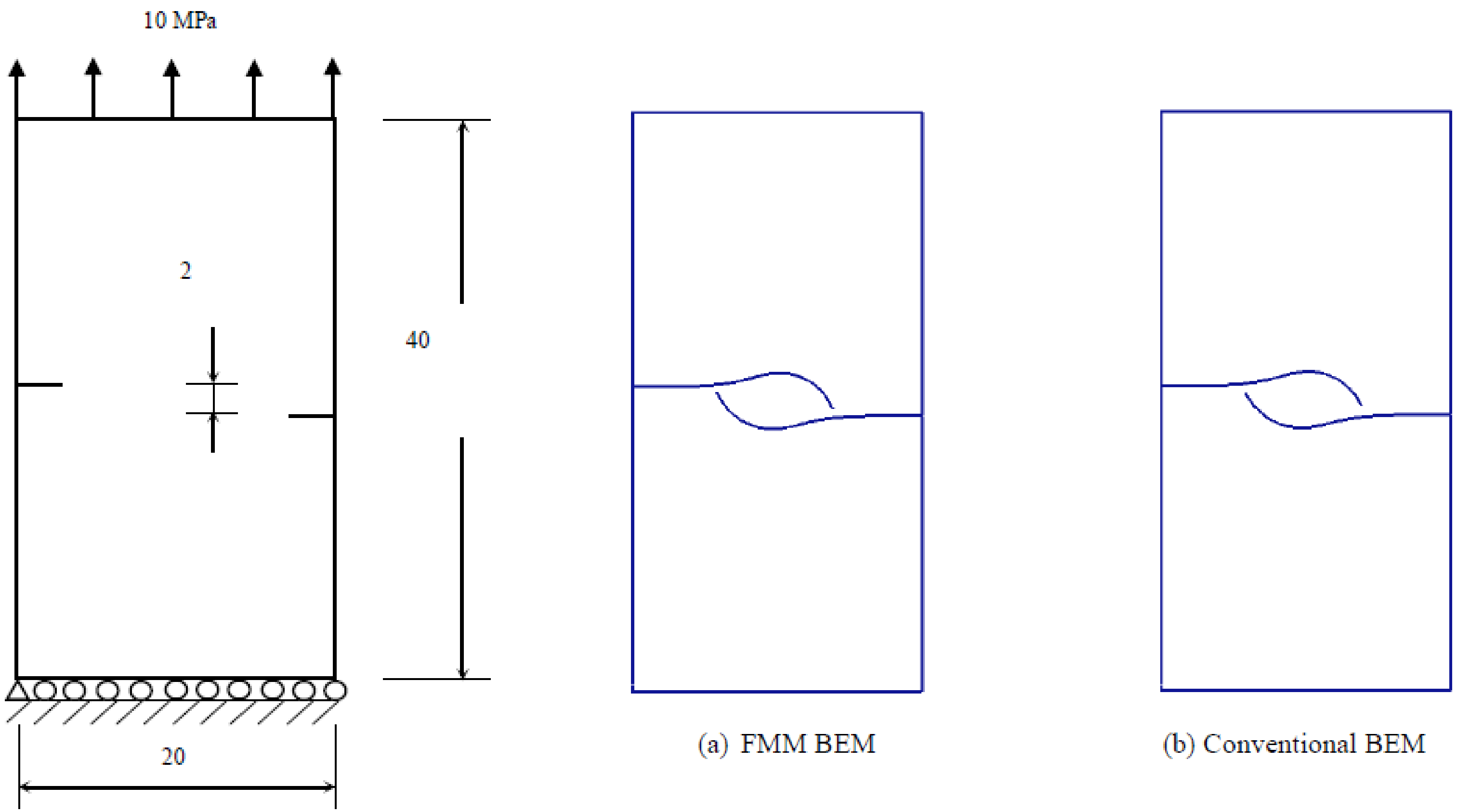
(a) FMM BEM



(b) Conventional BEM

A plate with an inclined center crack.

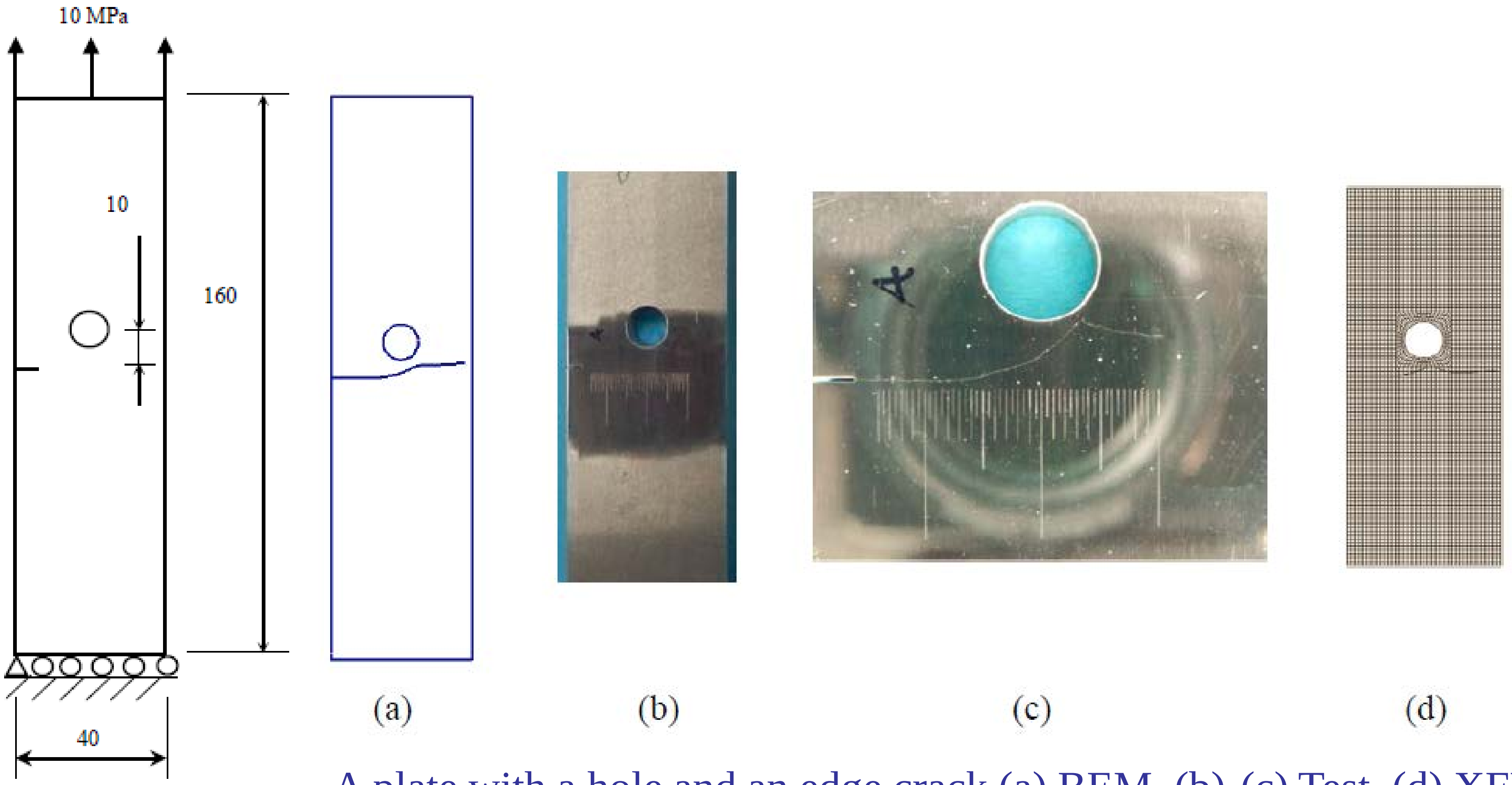
2-D Example: A Benchmark Problem



(a) FMM BEM (b) Conventional BEM

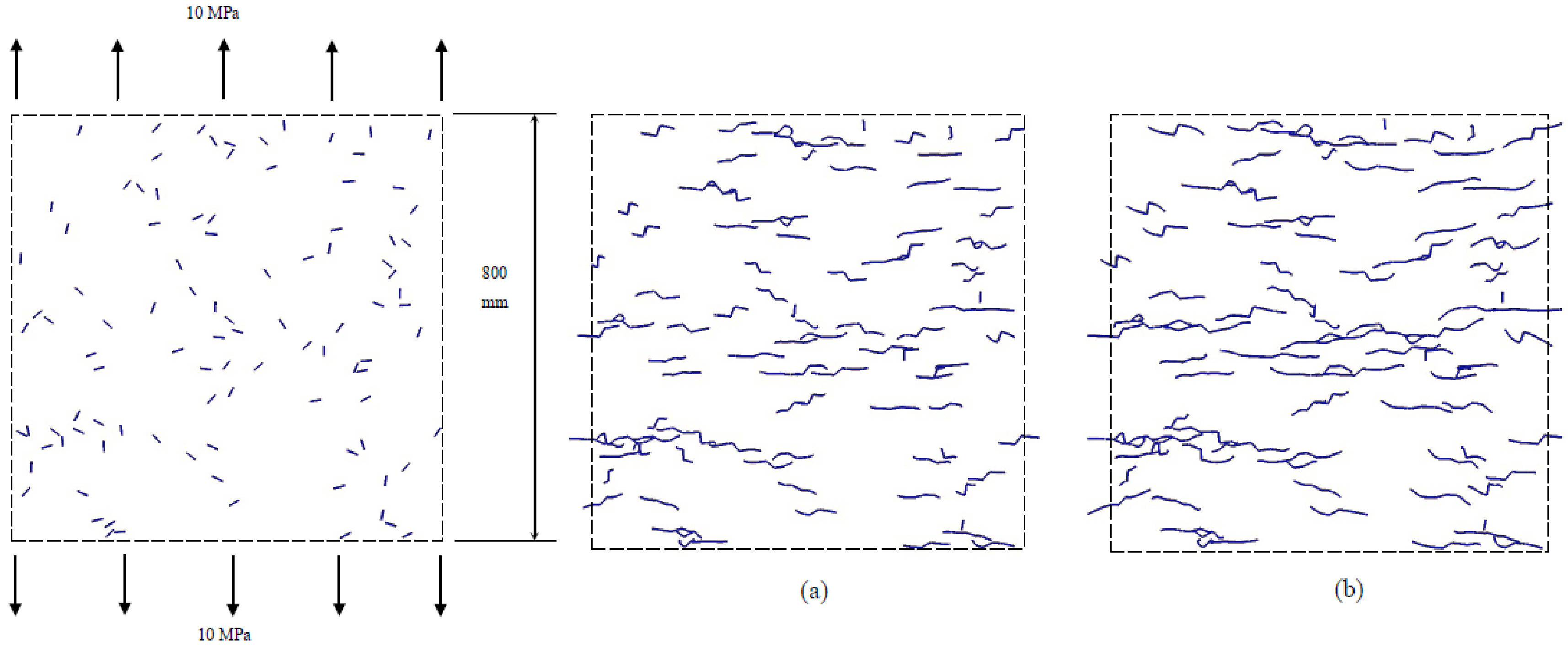
A plate with two edge cracks.

2-D Example: A Validation Problem



A plate with a hole and an edge crack (a) BEM, (b)-(c) Test, (d) XFEM.

Propagation of Multiple Cracks

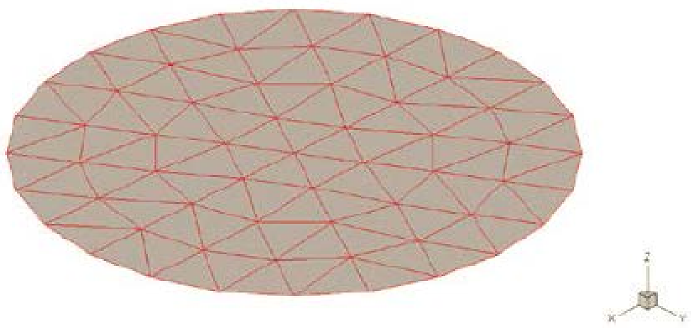


References:

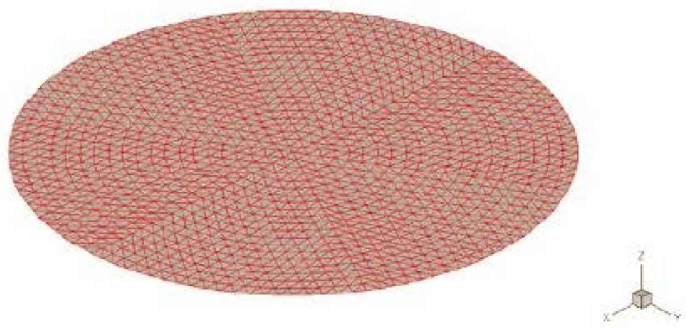
- Y. J. Liu, Y. X. Li, and W. Xie, "Modeling of multiple crack propagation in 2-D elastic solids by the fast multipole boundary element method," *Engineering Fracture Mechanics*, **172**, 1-16 (2017).

- Y. J. Liu, "On the displacement discontinuity method and the boundary element method for solving 3-D crack problems," *Engineering Fracture Mechanics*, **164**, 35-45 (2016).

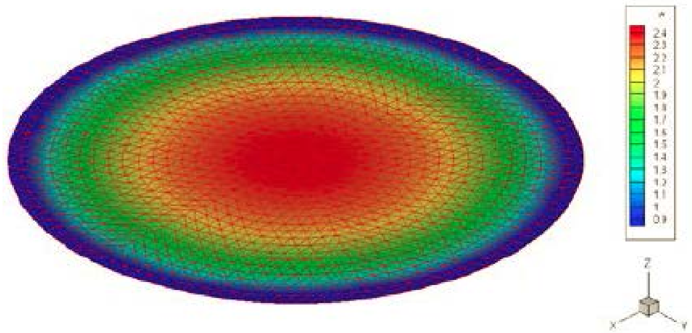
3-D Example: A Penny-Shaped Crack



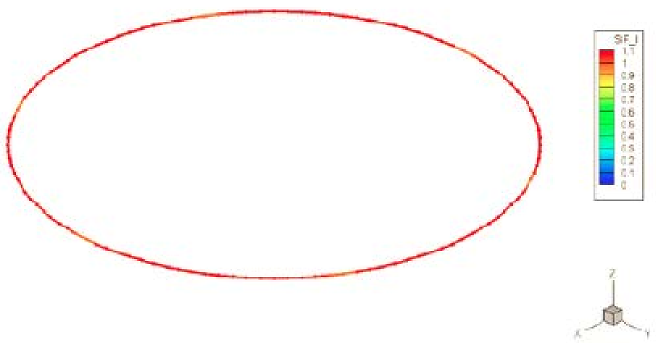
$M = 96$



$M = 2400$

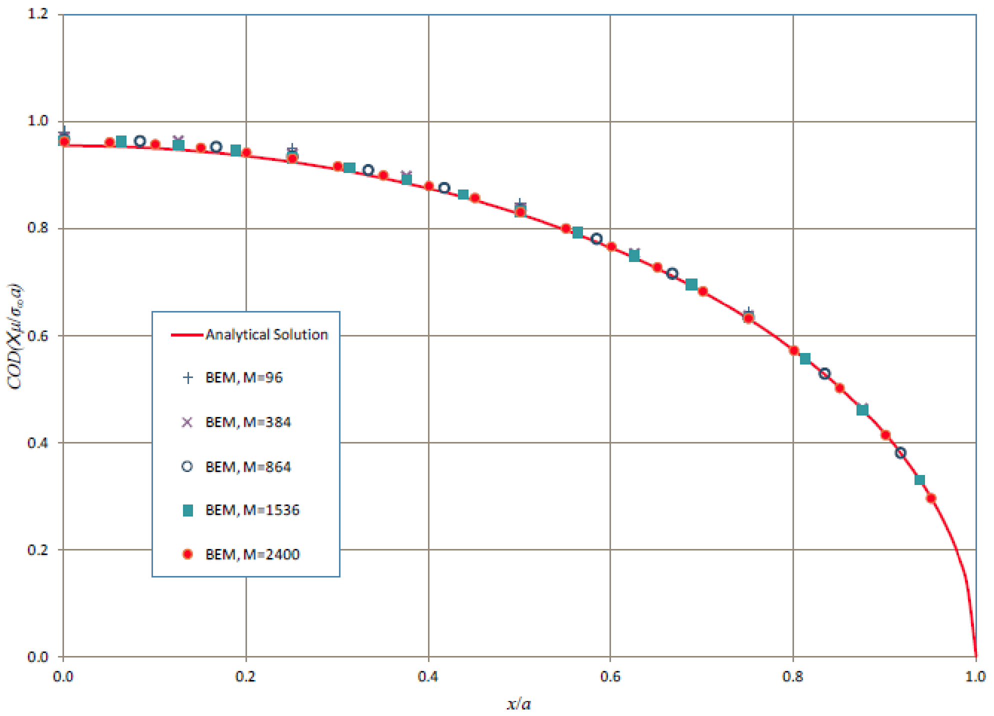


COD plot

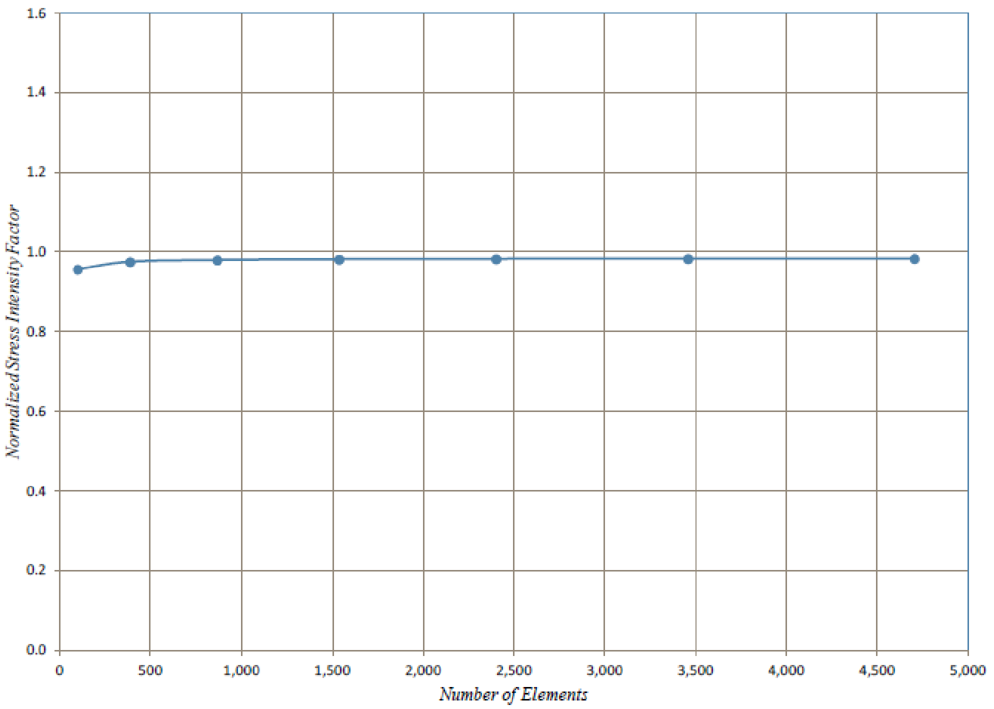


SIF plot

3-D Example: A Penny-Shaped Crack (Cont.)



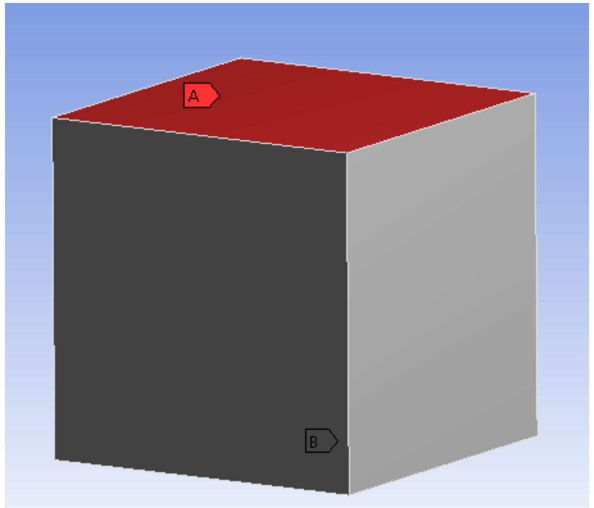
COD plot

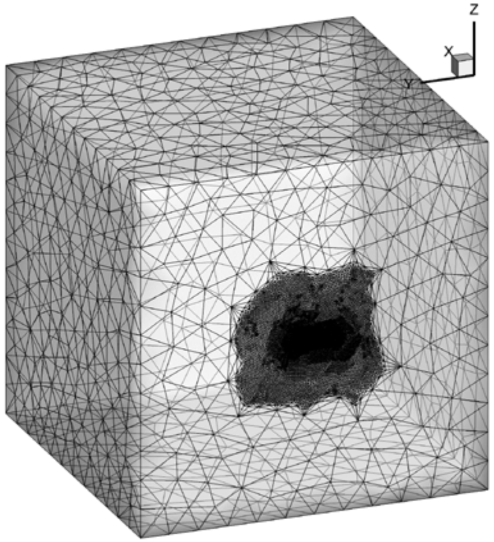


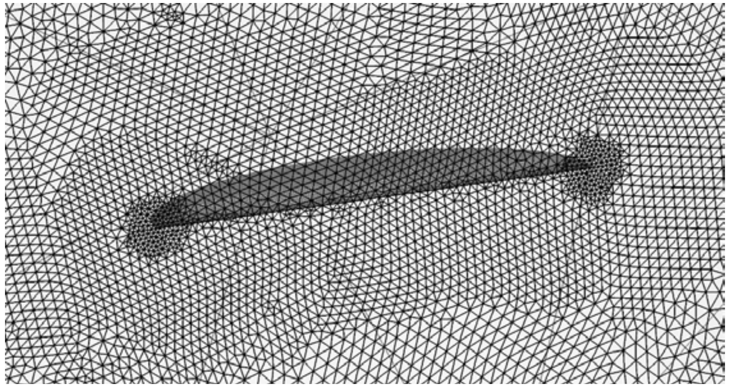
SIF plot

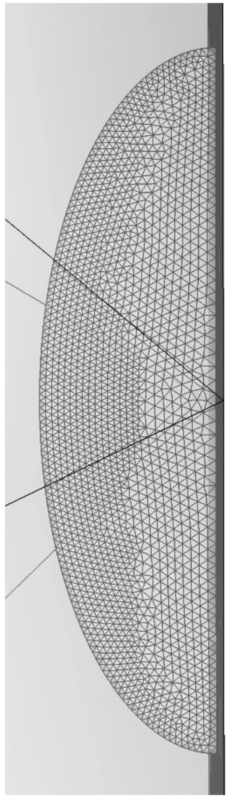
Semi-elliptical Surface Crack in A Block

- Surface crack in a cubic
  - Side length = 100 mm;
  - Major axis length = 5 mm;
  - Minor axis length = 2.5 mm.
- Load
  - Fixed on one end;
  - Traction load = 1 MPa on other.
- Material
  - E = 1 MPa;
  - Poisson’s ratio = 0.25.



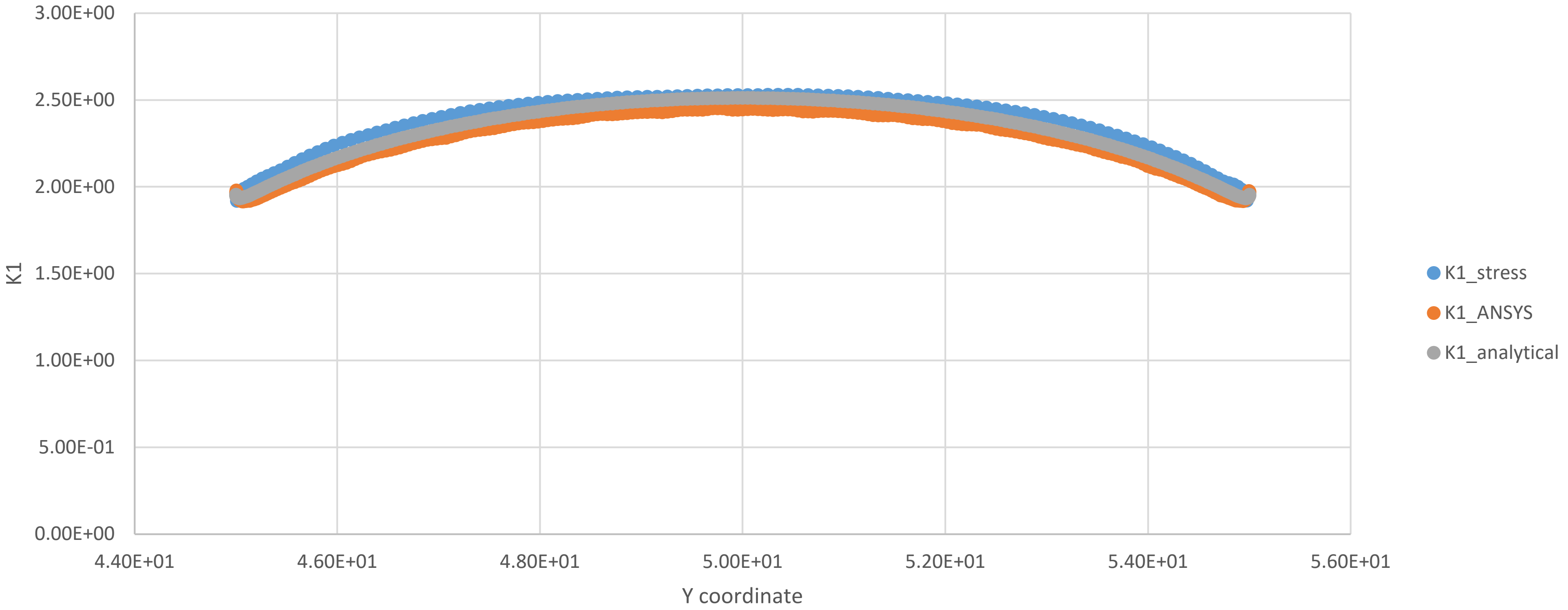






Semi-elliptical Surface Crack in A Block

- Results from 2nd layers of nodes
  - SIF of analytical solution is from reference [Anderson, 2005];
  - SIF is calculated from FEM using a nodal-force method where no prior assumption of plane stress or plane strain is required [Raju, 1979].

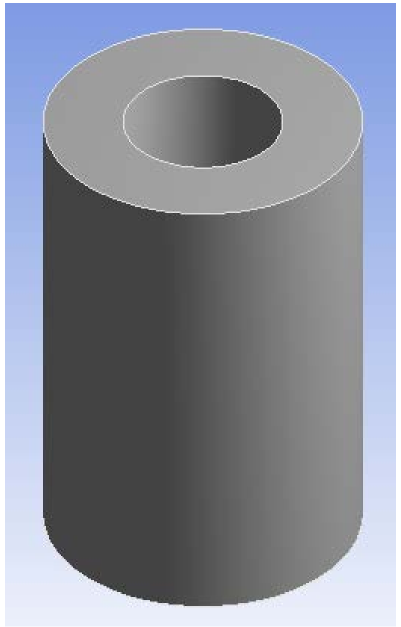


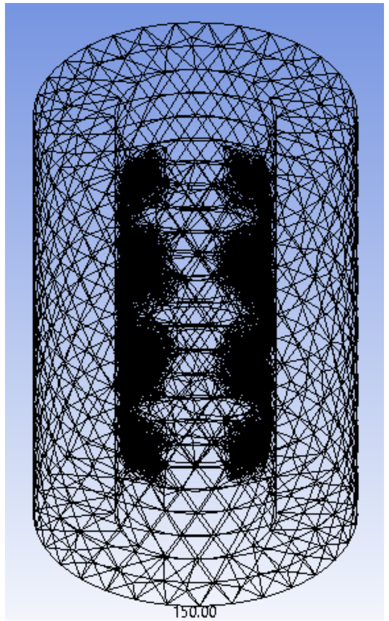
Semi-elliptical Surface Crack in A Block

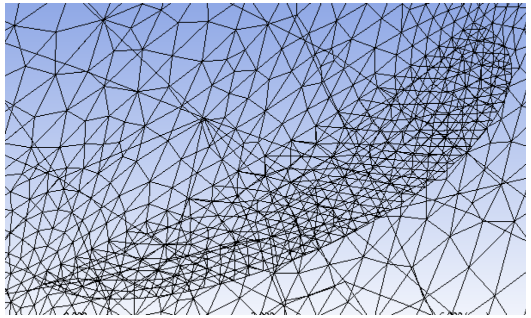
	No. Elements	No. Nodes	CPU time (s)	Elapsed time (s)
FEM (*ANSYS*)	5,068,305	6,802,492	9709.47	12,224.00
BEM	27,204	13,604	1778.20	264.10

Multiple Semi-elliptical Surface Cracks in A Hollow Cylinder

- Surface crack in a hollow cylinder
  - External radius = 100 mm;
  - Internal radius = 50mm;
  - Cracks
    - Major axis length = 5 mm;
    - Minor axis length = 2.5 mm;
    - 12 cracks evenly distributed along hollow cylinder
- Load
  - Fixed on one end;
  - Traction load = 1 MPa on other.
- Material
  - *E* = 1 MPa;
  - Poisson’s ratio = 0.25.

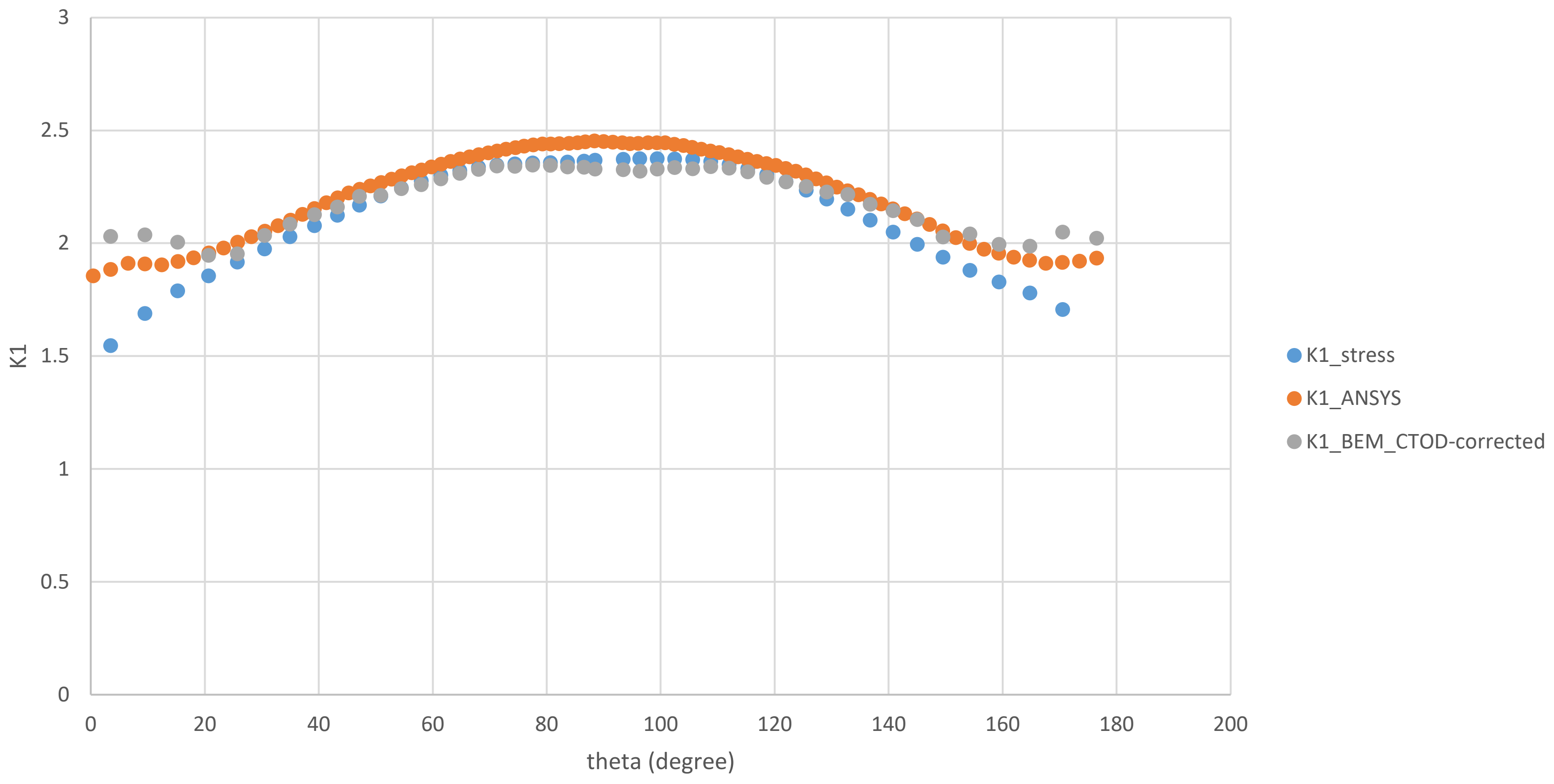


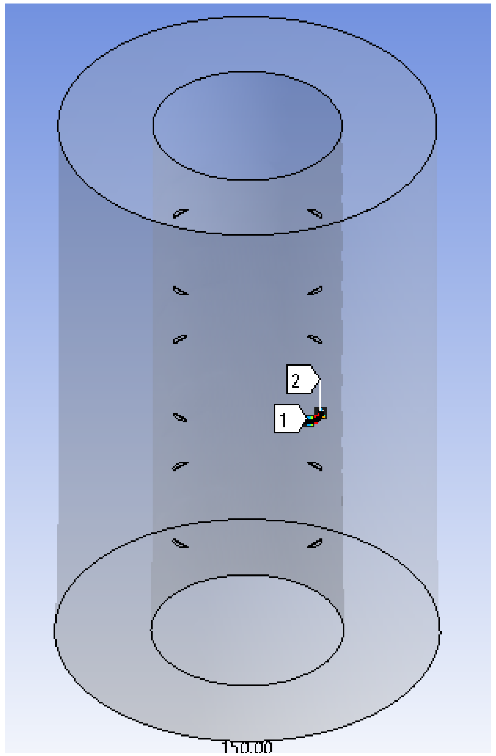




Multiple Semi-elliptical Surface Cracks in A Hollow Cylinder

Results: K_I along the crack front



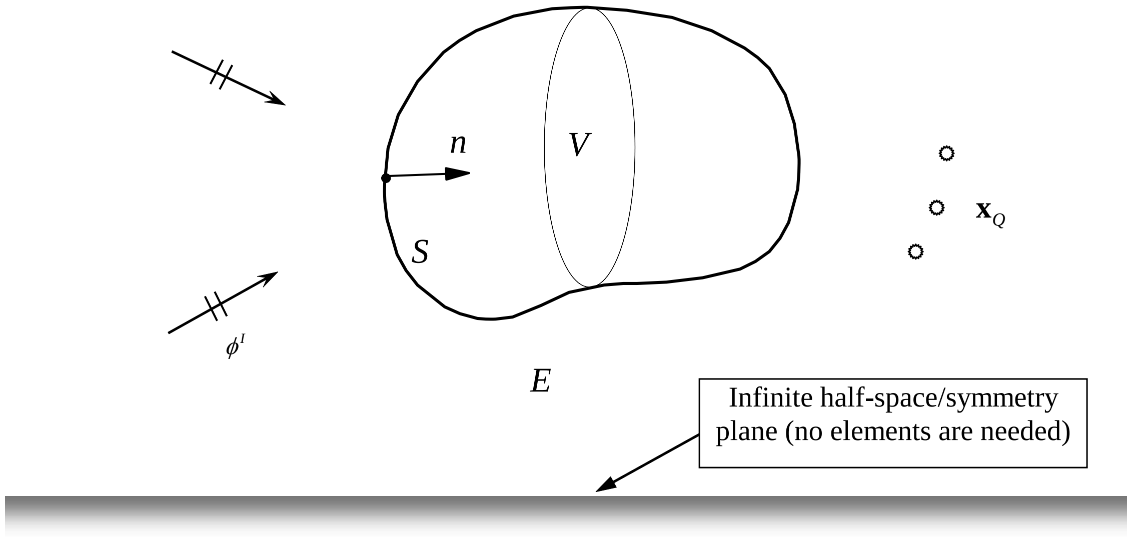


K_I of the highlighted crack is used for comparing the results

Multiple Semi-elliptical Surface Cracks in A Hollow Cylinder

	No. Elements	No. Nodes	CPU time (s)	Elapsed time (s)
FEM (*ANSYS*)	1,945,990	2,649,016	5356.05	8274.00
BEM	42,780	128,340	6865.09	958.97

Modeling Acoustic Wave Problems

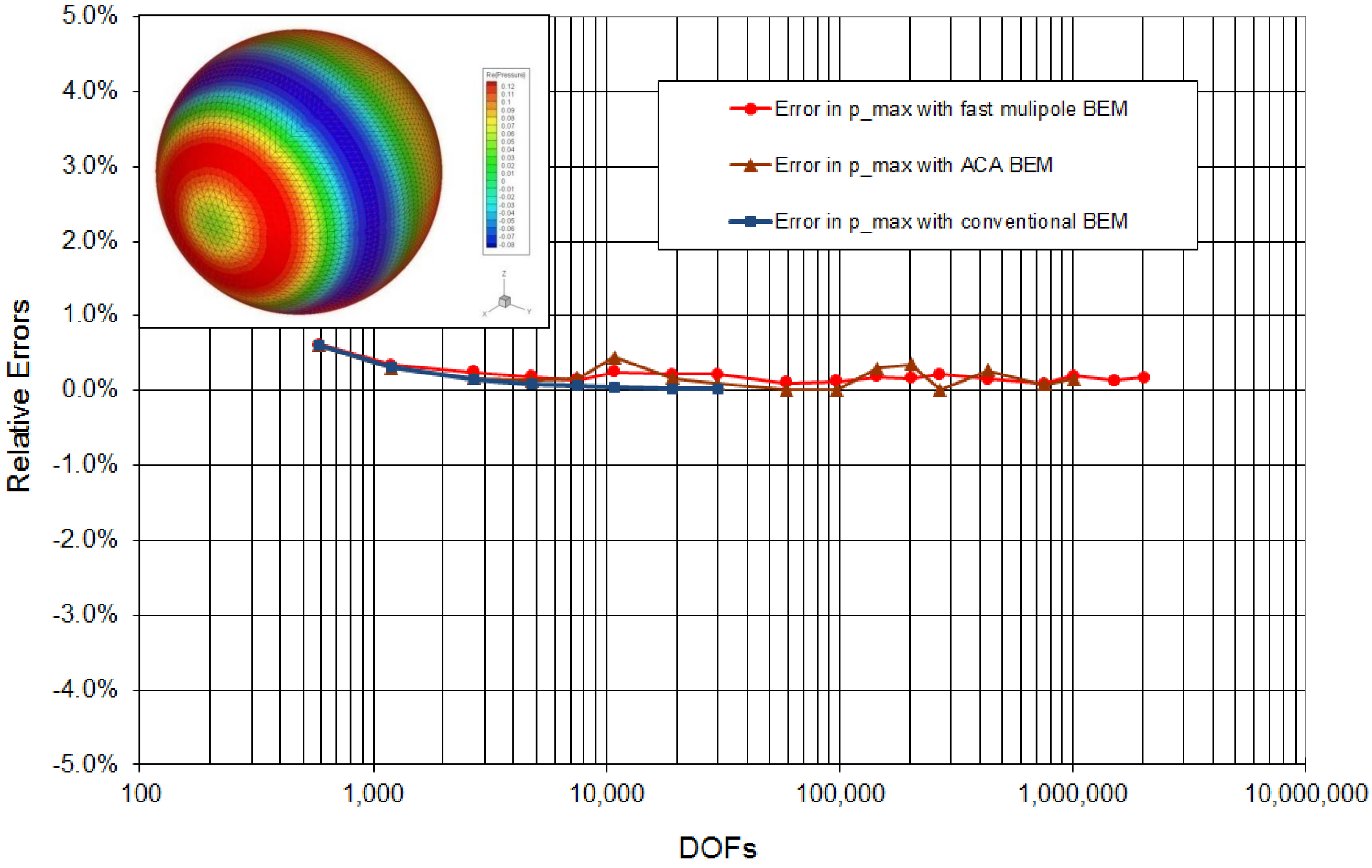


- Helmholtz equation:

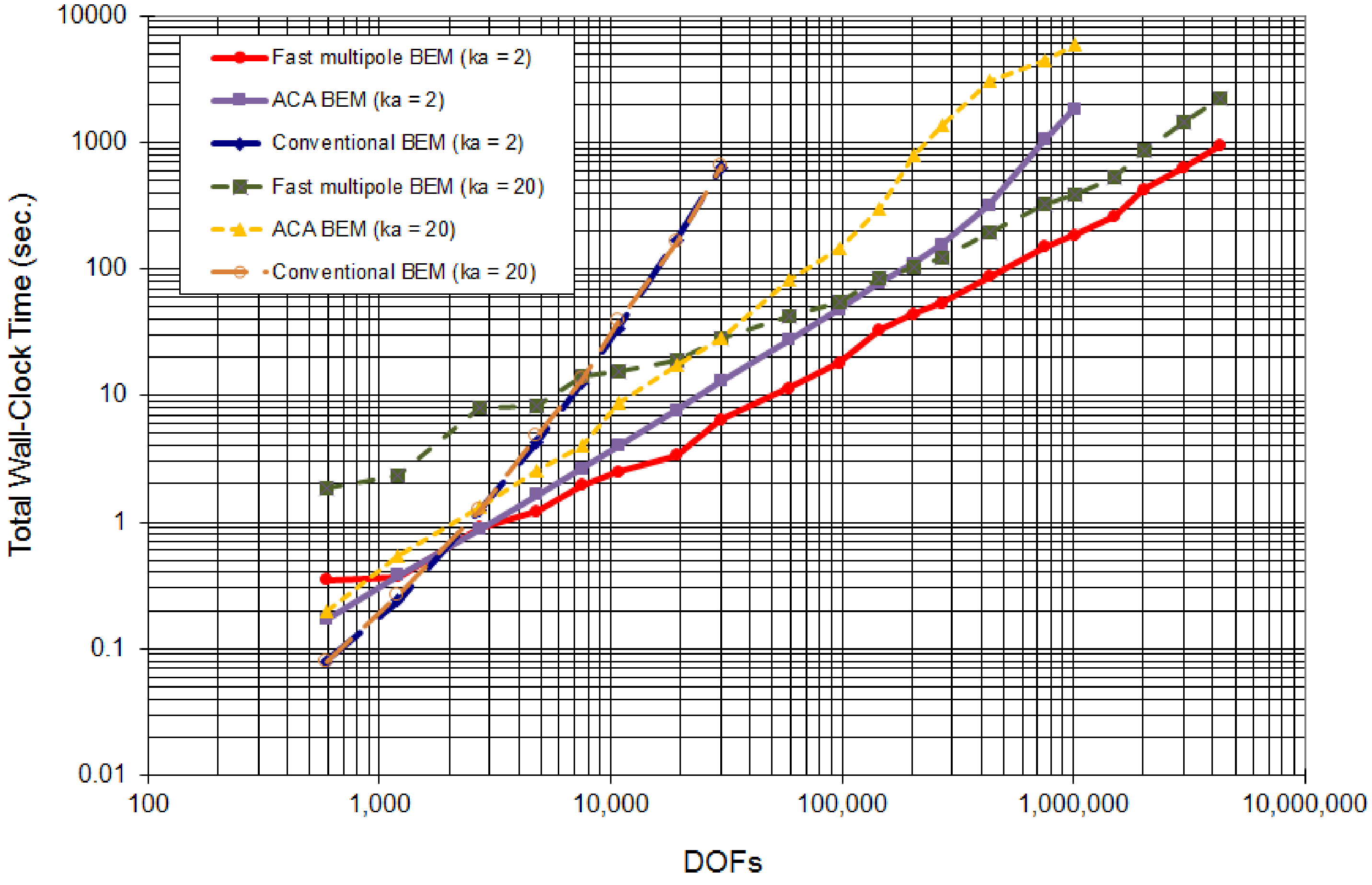
$$\nabla^2\phi + k^2\phi + Q\delta(\mathbf{x}, \mathbf{x}_Q) = 0, \qquad \forall \mathbf{x} \in E$$

- ϕ - acoustic pressure, $k = \omega / c$ - wavenumber
- BEM for solving 3-D full-/half-space, interior/exterior, radiation/scattering problems

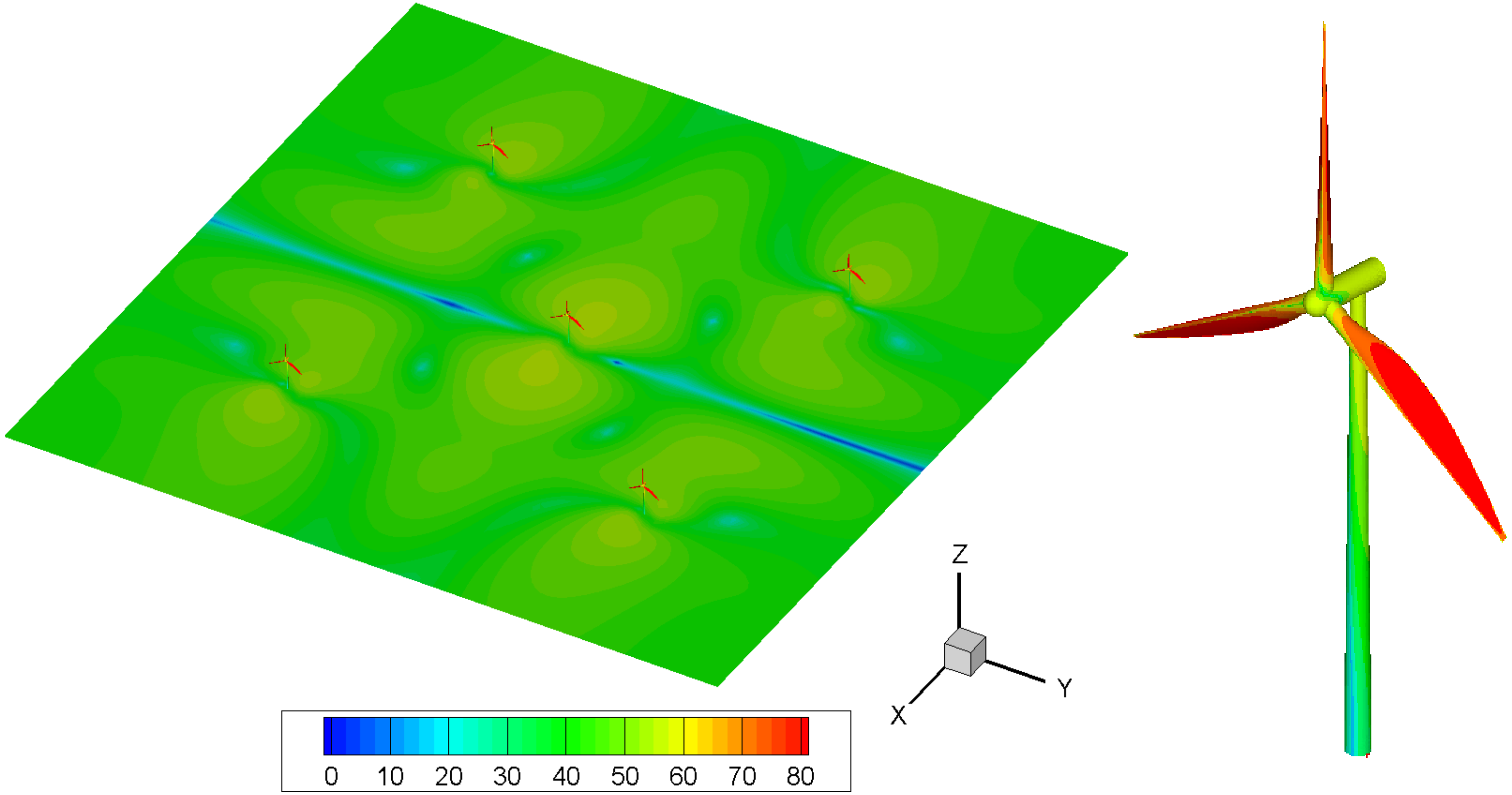
Examples: A Radiating Sphere



$O(N)$ Computing Efficiencies

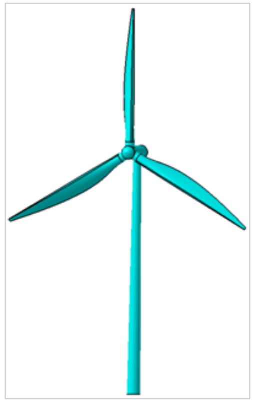


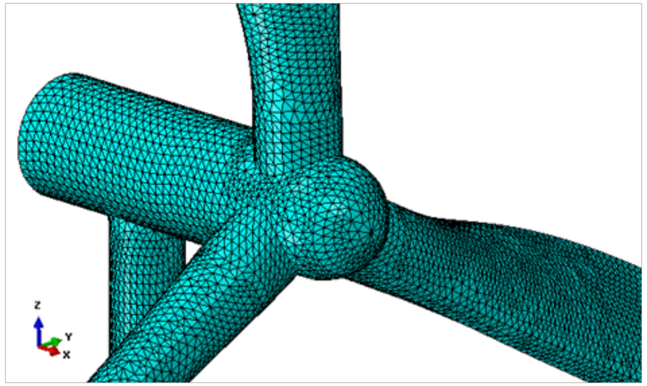
Windmill Turbine Analysis

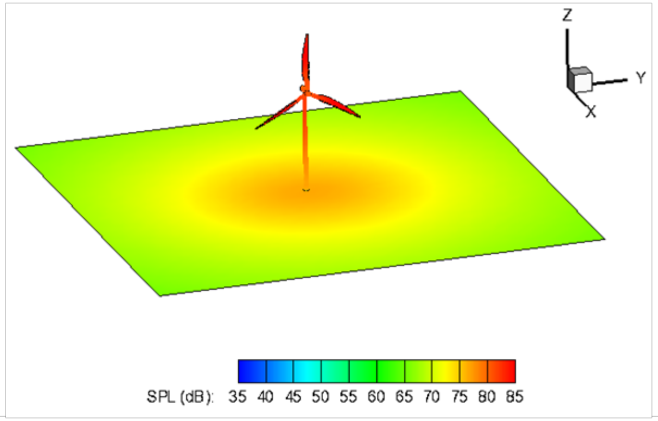


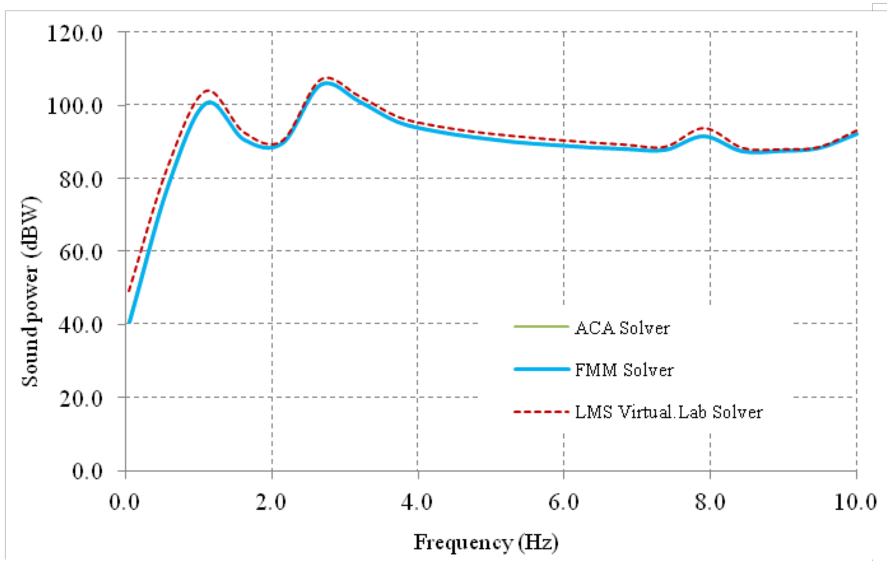
Plot of the SPL on the field due to 5 windmills (with 557,470 DOFs)

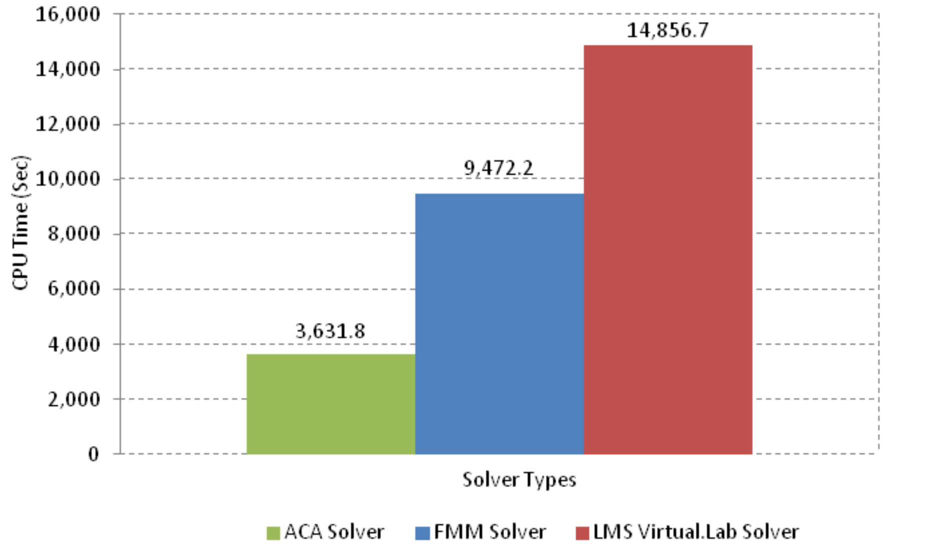
FEM/BEM Coupled Analysis (Freq. Response)



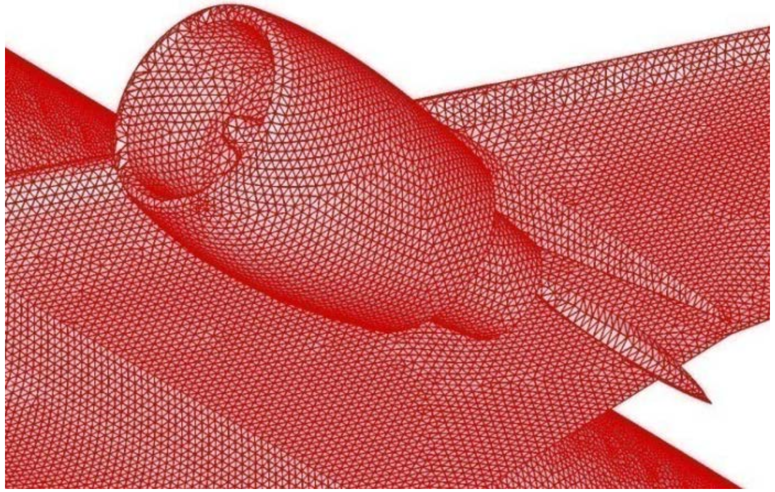




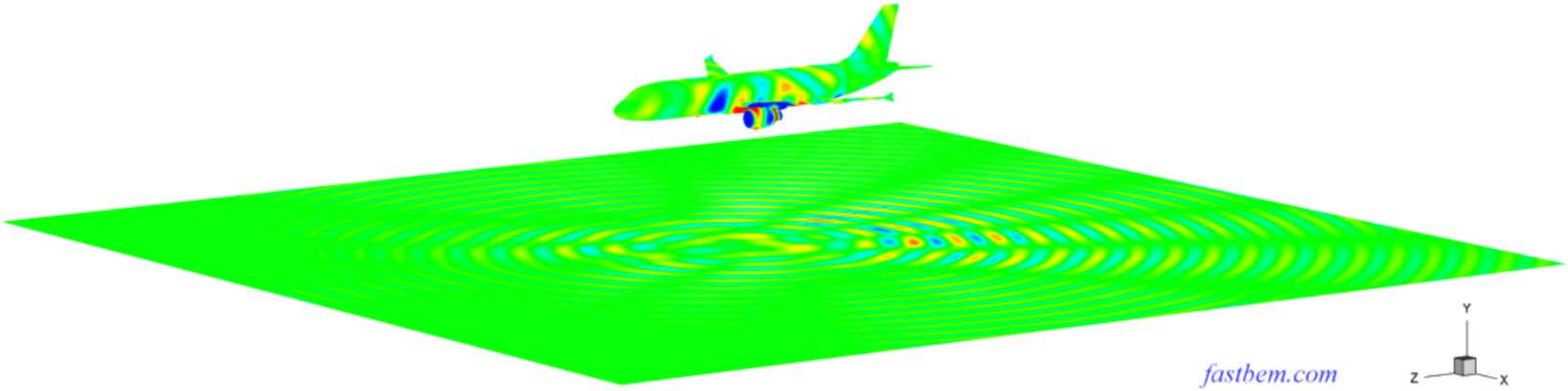




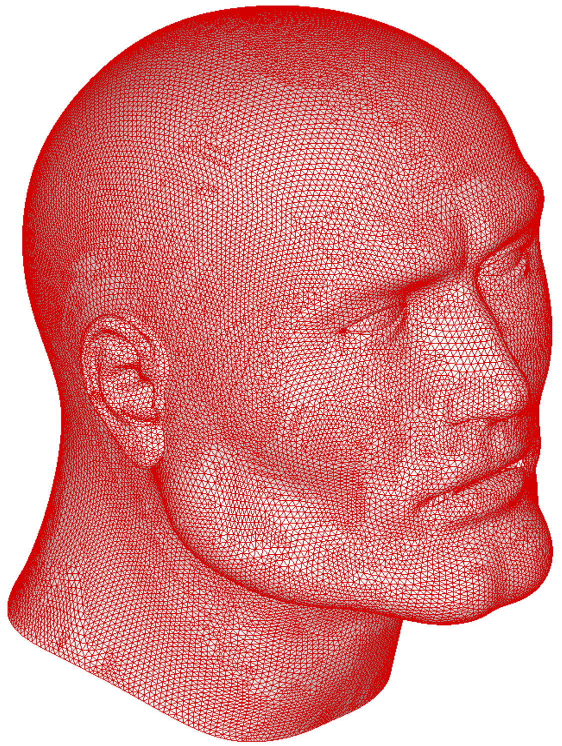
Noise Prediction in Airplane Landing/Taking Off



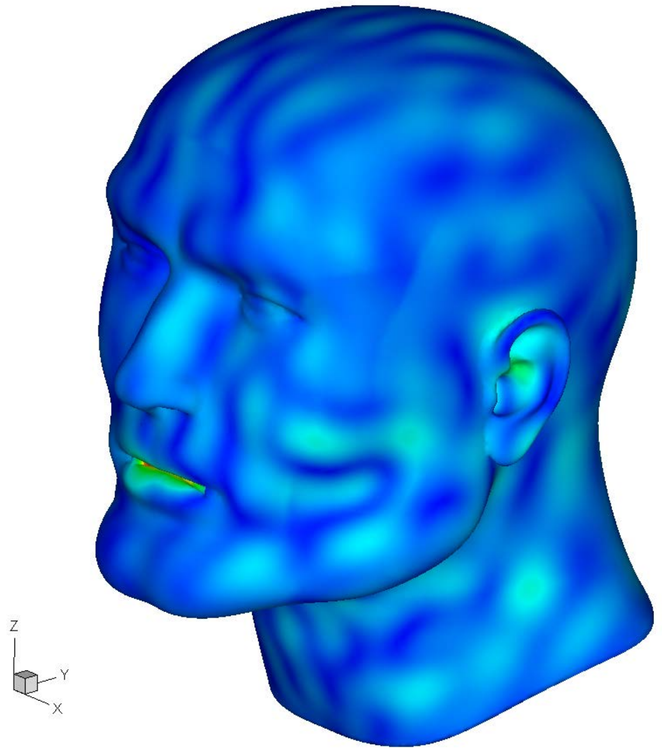
Noise propagation on the ground during the landing of an airplane, BEM model with 539,722 elements and solved with the FMM BEM in 8940 sec on a PC (ka = 61.5 or f = 90 Hz).



Bio-Medical Applications

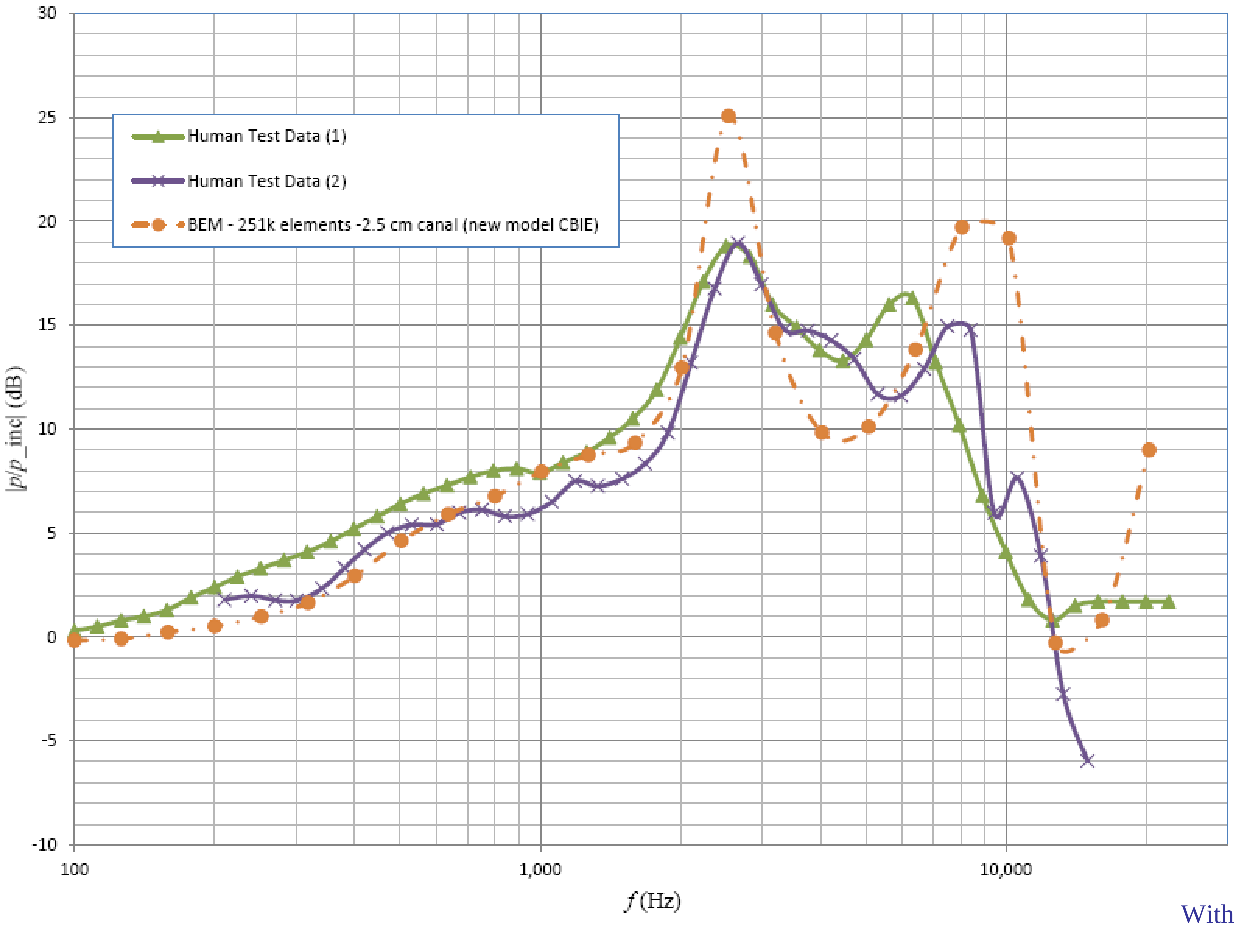


A human head model
with 90,000 elements



Pressure plots at 11 kHz
with a plane wave in –x direction

Bio-Medical Applications (Cont.)

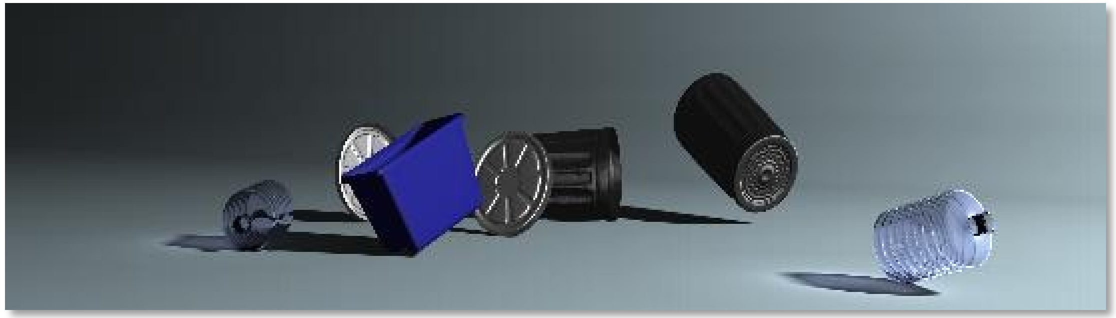


With Dr. J. Kim at UC

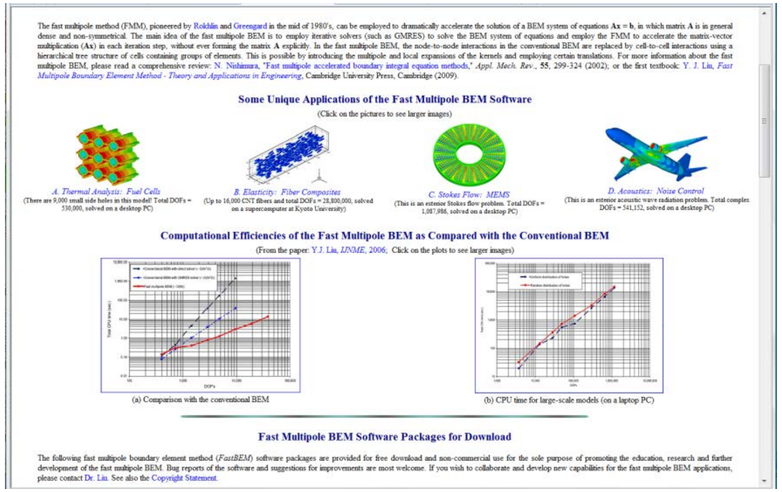
Applications in Computer Animation

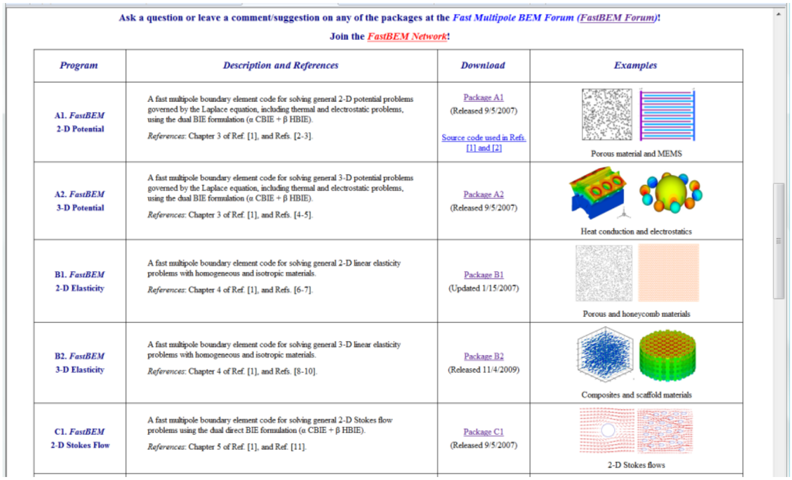
Work done by the Group of Professor Doug James at Cornell University, Using the *FastBEM Acoustics* code

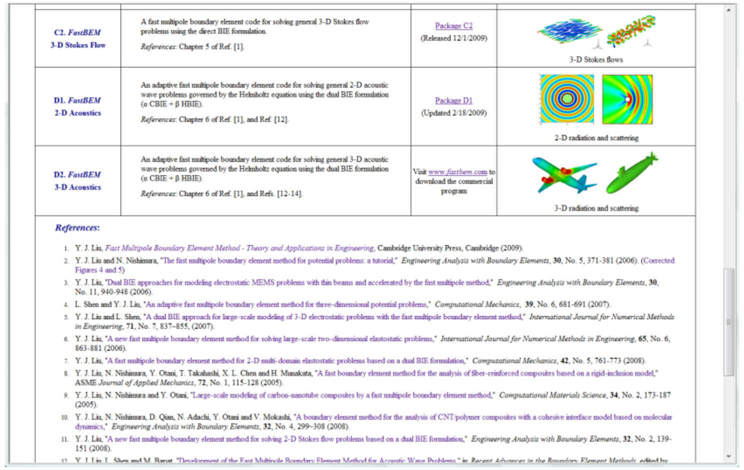
(Click on the images to play the YouTube video and *hear* the *computed* sound)



Fast Multipole Boundary Element Method (*FastBEM*) Software for Education, Research and Further Development (http://yijunliu.com/Software)





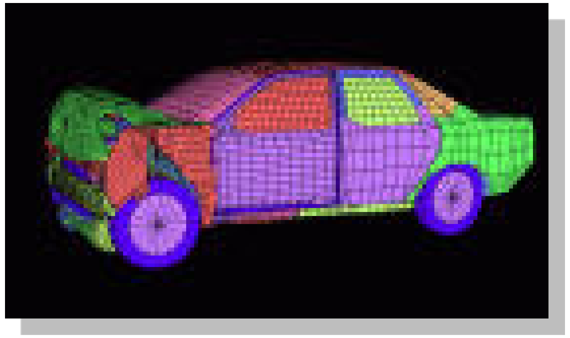


Summary

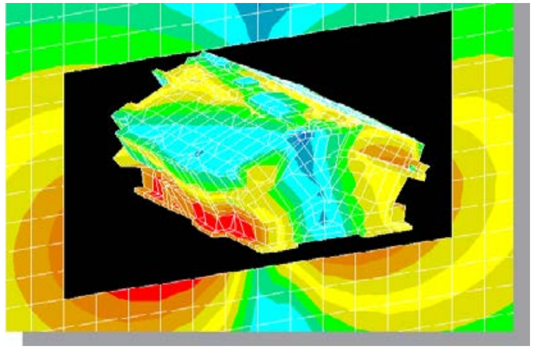
- BEM is very efficient for solving large-scale problems with complicated geometries or in infinite domains.
- Fast multipole method has re-energized the BEM research and dramatically expanded its range of applications.
- More large-scale, realistic engineering problems can be, and should be, solved by the fast multipole BEM.
- Other developments in fast multipole BEM: fracture mechanics, elastodynamic and electromagnetic wave propagation problems, time-domain problems, black-box fast multipole method (bbFMM), coupled field and nonlinear problems.
- Other fast solution methods for solving BIE/BEM equations include: adaptive cross approximation (ACA) method, precorrected FFT method, wavelet method, and others.

A Bigger Picture of the CM
– A Numerical Toolbox

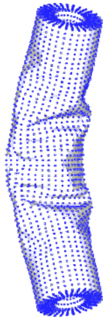
FEM: Large-scale structural, nonlinear, and transient problems



BEM: Large-scale continuum, linear, and steady state (wave) problems



Meshfree: Large deformation, fracture and moving boundary problems

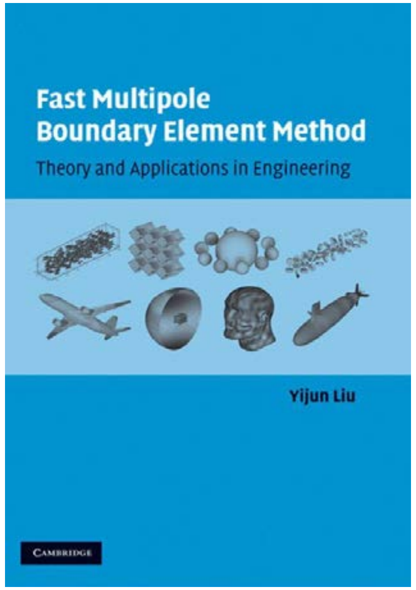


"If the only tool you have is a hammer, then every problem you can solve looks like a nail!"

References

1. L. F. Greengard, *The Rapid Evaluation of Potential Fields in Particle Systems* (The MIT Press, Cambridge, 1988).
2. N. Nishimura, "Fast multipole accelerated boundary integral equation methods," ASME *Applied Mechanics Reviews*, **55**, No. 4 (July), 299-324 (2002).
3. Y. J. Liu, S. Mukherjee, N. Nishimura, M. Schanz, W. Ye, A. Sutradhar, E. Pan, N. A. Dumont, A. Frangi and A. Saez, "Recent advances and emerging applications of the boundary element method," ASME *Applied Mechanics Review*, **64**, No. 5 (May), 1–38 (2011).
4. Y. J. Liu, *Fast Multipole Boundary Element Method - Theory and Applications in Engineering* (Cambridge University Press, Cambridge, 2009).
5. Y. J. Liu and N. Nishimura, "The fast multipole boundary element method for potential problems: a tutorial," *Engineering Analysis with Boundary Elements*, **30**, No. 5, 371-381 (2006).
6. Y. J. Liu, Fast Multipole Boundary Element Method (*FastBEM*) Software for Education, Research and Further Development (1997-2018), http://www.yijunliu.com/Software/

   (or **Google** search "**fast multipole BEM**")



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Contact

Dr. Yijun Liu
Chair Professor
Department of Mechanics and Aerospace Engineering
Southern University of Science and Technology (SUSTech)
Shenzhen, China

E-mail: liuyj3@sustech.edu.cn
Web: www.yijunliu.com