Finite deflection analysis of elastic plate by the boundary element method

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An integral equation formulation for finite deflection analysis of thin elastic plates is presented, based on general nonlinear differential equations which are equivalent to the von Kármán equations and by virtue of generalized Green identities. Boundary element discretization is applied and a relaxation iterative approach is employed to solve the nonlinear plate bending problems. A number of numerical examples are given; the results of computation are compared with the analytical solutions and good agreement is observed. It appears that the approach developed in this paper is effective.

Key words: plate, finite deflection, BEM

Much progress has been made in plate bending analysis by the boundary element method (BEM). The rapid development of this method stems from the fact that a reduction in dimensionality can often be accomplished which may result in a significant decrease in computational effort. For the linear analysis of thin plates, various integral equation formulations have been established and numerical solutions have been obtained.[1-9] The applications of BEM to plate analysis have also spread to the realms of free vibration and instability analysis,[10] time-dependent inelastic analysis of transverse deflection,[11] and Reissner's plate model.[12] The authors, however, know of little work which has been done on the finite deflection analysis of elastic plates by BEM. Tanaka[13] and Kamiya[14-15] are known to have published work in this field. Tanaka[13] studied the application of BEM to elastic plate bending problems with large deflections. He presented incremental integral equation formulations, which are equivalent to the von Kármán equations, while Kamiya investigated the large deflection of elastic plates based on Berger's equation.[16] The theoretical validity of Berger's equation is often questioned and its practical use is confined to such boundary conditions as the in-plane displacements being constrained on the boundary. Recently, on the basis of the weighted residual method, Kamiya and Sawaki have presented the formulation of finite deflection analysis of plate by virtue of von Kármán plate equations,[17] but no numerical results have been given.

In this paper a complete formulation is presented, which is different from Tanaka's[13] and Kamiya's,[17] for the finite deflection analysis of thin elastic plates by the BEM. Starting with the general nonlinear differential equations of finite deflection of plates, which are equivalent

to the von Kármán equations (see, for example, formulae (1.60) and (1.61) in reference 18, p. 36), the BIE formulation can be deduced by means of the generalized Green identities. These integral equations possess the ability to solve the large deflection problem of plates under arbitrary boundary conditions (e.g. clamped, simply-supported, free, in-plane constrained or unconstrained) and different load conditions (e.g. transverse load, in-plane load or a combination of these). Under certain boundary conditions, some difficulties may occur in the direct use of von Kármán equations. In the formulations presented here, the effect of the interaction of the bending and membrane strains is included in the nonlinear coupling terms. An iterative procedure is applied to achieve the linearization of the nonlinear equations. In the iteration process, the proceeding results are substituted into the nonlinear coupling terms to obtain the linearized formulations for the current calculation. By virtue of discretization, the boundary integral equations are transformed into two sets of algebraic equations which correspond to the bending and membrane actions respectively. A relaxation factor has been introduced in the computation to accelerate the convergence of the iteration process. Numerical examples show that the approach developed in this paper is effective to be employed to solve the finite deflection problems of plate with various boundary conditions and load conditions.

Basic relationships

The theory of finite deflection of the plate is developed in detail in the literature (e.g. reference 18). Basic relationships are outlined here. Consider a thin elastic plate with x_1- and x_2-coordinate axes corresponding to the planar

middle surface of the plate, *Figure 1a*. An infinitesimal element of the plate, subjected to stresses and loads, is portrayed in *Figure 1b*. The material constants of the plate are represented by E (Young's modulus) and ν (Poisson's ratio) and the bending rigidity $D = Eh^3/12(1-\nu^2)$, where h is the thickness of the plate. The fundamental non-linear differential equations for large deflection of the plate, which are equivalent to von Kármán equations, are:

$$D\nabla^4 w = \bar{q} + h\sigma_{ij}w_{,ij} \tag{1}$$

$$\sigma_{ij,j} = 0 \tag{2}$$

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + w_{,i}w_{,j}) \tag{3}$$

$$\sigma_{ij} = E_{ijkl}e_{kl} \tag{4}$$

where the repeated indices imply the Einstein summation convention with the indices $i, j, k, l, \ldots, \in\{1, 2\}$. The tensor:

$$E_{ijkl} = \frac{2G\bar{\nu}}{1 - 2\bar{\nu}}\,\delta_{ij}\,\delta_{kl} + G(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})$$

where $\bar{\nu} = \nu/(1+\nu)$, $G = E/2(1+\nu)$ and δ_{ij} is the Kronecker symbol.

The bending and twisting moments M_{ij} can be evaluated by the following relationships in terms of the deflection w:

$$\left.\begin{aligned}
M_{11} &= -D(w_{,11} + \nu w_{,22}) \\
M_{22} &= -D(w_{,22} + \nu w_{,11}) \\
M_{12} &= M_{21} = -D(1-\nu)\,w_{,12}
\end{aligned}\right\} \tag{5}$$

The bending and twisting moments on the boundary Γ can be written as:

$$\left.\begin{aligned}
M_n &= M_{11}n_1^2 + M_{22}n_2^2 + 2M_{12}n_1n_2 \\
M_{nt} &= (M_{22} - M_{11})n_1n_2 + M_{12}(n_1^2 - n_2^2)
\end{aligned}\right\} \tag{6}$$

where $n_i = \cos(n, x_i)$ are direction cosines of the outward normal n at the boundary.

The shear forces:

$$Q_i = M_{ij,j} \tag{7}$$

On the boundary, the shear force Q_n and the Kirchhoff equivalent shear force K_n can be expressed as:

(1) linear case:

$$\left.\begin{aligned}
Q_n &= Q_k n_k \\
K_n &= Q_n + M_{nt,s}
\end{aligned}\right\} \tag{8}$$

(2) nonlinear case:

$$\left.\begin{aligned}
Q_n^{(NL)} &= Q_k n_k + p_k w_{,k}h \\
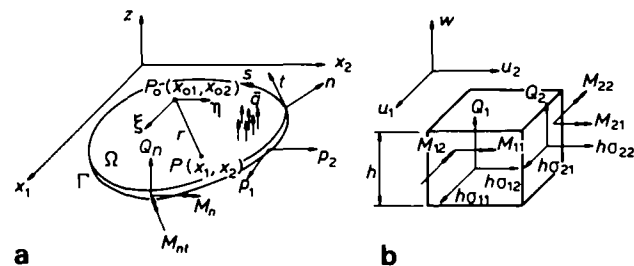K_n^{(NL)} &= Q_n^{(NL)} + M_{nt,s}
\end{aligned}\right\} \tag{9}$$

a   **b**

Figure 1 Notation

The boundary conditions are:

(1) geometrical boundary conditions:

$$w = \bar{w} \qquad w_{,n} = \overline{w_{,n}} \tag{10}$$

$$u_i = \bar{u}_i \tag{11}$$

(2) mechanical boundary conditions:

$$M_n = \bar{M}_n \qquad K_n^{(NL)} = \bar{K}_n^{(NL)} \tag{12}$$

$$p_i = \sigma_{ij}n_j = \bar{p}_i \tag{13}$$

where $(\bar{\ })$ means that the value is prescribed on the boundary.

Integral equation formulation

If the left-hand side of equation (1) is defined by:

$$q = \bar{q} + h\sigma_{ij}w_{,ij}$$

a pseudo-transverse distributed load exists. It implies that the actual load $\bar{q}$ is adjusted by the corrector $h\sigma_{ij}w_{,ij}$ due to the nonlinearity of large deflection of plates. Thus, equation (1) is of the form:

$$D\nabla^4 w = q \tag{14}$$

The fundamental solution of biharmonic equation is:

$$w^* = w^*(P, P_0) = \frac{1}{8\pi}r^2 \ln r \tag{15}$$

where $r = |\overrightarrow{P_0 P}|$, see *Figure 1a*.

Substitution of equation (15) into expressions (6) and (8) gives the bending moment M_n^* and the equivalent shear force K_n^* corresponding to the fundamental solution w^*.

According to equation (14), by means of the Rayleigh–Green identity,[6] the integral equation corresponding to the nonlinear bending deformation can be obtained in the form:

$$\int_\Gamma [w^*K_n - wK_n^* + w_{,n}M_n^* - w_{,n}^*M_n]\,d\Gamma + \int_\Omega qw^*\,d\Omega$$

$$= \begin{cases} Dw(P_0) & P_0 \in \Omega \tag{16} \\ \tfrac{1}{2}Dw(P_0) & P_0 \in \Gamma \tag{17} \end{cases}$$

where the boundary Γ is supposed to be smooth enough in the sense of Lyapunov.

Let ξ be a vector at a point P_0 in the domain Ω of the plate (see *Figure 1a*). But, if the point P_0 is on the boundary Γ, ξ denotes the outward normal. Differentiating both sides of equations (16) and (17) with respect to ξ, one obtains:

$$\int_\Gamma [w_{,\xi}^*K_n - wK_{n,\xi}^* + w_{,n}M_{n,\xi}^* - w_{,n\xi}^*M_n]\,d\Gamma$$

$$+ \int_\Omega qw_{,\xi}^*\,d\Omega = \begin{cases} Dw_{,\xi}(P_0) & P_0 \in \Omega \tag{18} \\ \tfrac{1}{2}Dw_{,\xi}(P_0) & P_0 \in \Gamma \tag{19} \end{cases}$$

Equations (17) and (19) are the boundary integral equations equivalent to the differential equation (1). Once the unknowns included in these equations have been determined the deflection and rotation at any interior point of the domain can be calculated by use of equations (16) and (18).

Differentiation of equation (18) with respect to η yields:

$$Dw_{,\xi\eta}(P_0) = \int_\Gamma [w^*_{,\xi\eta}K_n - wK^*_{n,\xi\eta} + w_{,n}M^*_{n,\xi\eta}$$

$$- w^*_{,n\xi\eta}M_n] \, d\Gamma + \int_\Omega qw^*_{,\xi\eta} \, d\Omega$$

$$P_0 \in \Omega \qquad (20)$$

where η is another vector at the point P_0.

Equation (20) can be applied to calculate the value of $w_{,ij}$ at any point in the domain Ω, then, substitution of these values into equation (5) provides the bending and twisting moments. The second set of boundary integral equations, which represents the nonlinear membrane deformation state, is established as follows: the fundamental solution of the two-dimensional elasticity equation is that which satisfies the equation:

$$\sigma^*_{jk,k} + \delta_{ij}(P,P_0) = 0 \qquad (21)$$

here $\delta_{ij}(P,P_0)$ is the Dirac δ-function which represents a component in the direction x_j of a unit force acting at point P_0 in the direction x_i.

In conformity with the linear fundamental solution, the strain–displacement and stress–strain relations can be written, respectively, as:

$$e^*_{ij} = \tfrac{1}{2}(u^*_{i,j} + u^*_{j,i}) \qquad (22)$$

$$\sigma^*_{ij} = E_{ijkl}e^*_{kl} \qquad (23)$$

As a consequence of the application of the Green formula, taking into account equations (3), (4), (22) and (23), the following identity can be established:

$$\int_\Omega \sigma_{ij,j}u^*_i \, d\Omega - \int_\Omega \sigma^*_{ij,j}u_i \, d\Omega$$

$$= \int_\Gamma p_iu^*_i \, d\Gamma - \int_\Gamma p^*_iu_i \, d\Gamma - \frac{1}{2}\int_\Omega \sigma^*_{ij}w_{,i}w_{,j} \, d\Omega \quad (24)$$

Substitution of equations (2) and (21) into (24) gives:

$$\int_\Gamma p_ku^*_{ik} \, d\Gamma - \int_\Gamma p^*_{ik}u_k \, d\Gamma - \frac{1}{2}\int_\Omega \sigma^*_{ikl}w_{,k}w_{,l} \, d\Omega$$

$$= \begin{cases} u_i(P_0) & P_0 \in \Omega \qquad (25) \\ \tfrac{1}{2}u_i(P_0) & P_0 \in \Gamma \qquad (26) \end{cases}$$

where the boundary Γ is smooth and the subscript i indicates that the direction of the unit force is x_i.

The fundamental solution u^*_{ik} and p^*_{ik} for the isotropic material is given in reference 19, p. 139, and the expression for σ^*_{ikl} is written as:

$$\sigma^*_{ikl} = -\frac{1}{4\pi(1-\bar\nu)r} [2r_{,i}r_{,k}r_{,l}$$

$$+ (1-2\bar\nu)(\delta_{ik}r_{,l} + \delta_{il}r_{,k} - \delta_{kl}r_{,i})] \qquad (27)$$

where $\bar\nu = \nu/(1+\nu), r_{,i} = \partial r/\partial x_i$.

Equation (26) is the boundary integral equation which is equivalent to the differential equations of nonlinear membrane deformation (2)–(4).

Differentiating both sides of equation (25), yields:

$$\frac{\partial u_i(P_0)}{\partial x_{0j}} = \int_\Gamma p_k \frac{\partial u^*_{ik}}{\partial x_{0j}} \, d\Gamma - \int_\Gamma \frac{\partial p^*_{ik}}{\partial x_{0j}} u_k \, d\Gamma$$

$$- \frac{1}{2}\frac{\partial}{\partial x_{0j}} \int_\Omega \sigma^*_{ikl}w_{,k}w_{,l} \, d\Omega \qquad P_0 \in \Omega \quad (28)$$

Considering that the expression (27) includes singularity, the derivative of the improper integral:

$$\int_\Omega \sigma^*_{ikl}w_{,k}w_{,l} \, d\Omega$$

with respect to the parameter x_{0j} can be performed as:

$$\frac{\partial}{\partial x_{0j}} \int_\Omega \sigma^*_{ikl}w_{,k}w_{,l} \, d\Omega$$

$$= \int_\Omega \frac{\partial \sigma^*_{ikl}}{\partial x_{0j}} w_{,k}w_{,l} \, d\Omega$$

$$+ \frac{1}{8(1-\bar\nu)} [2(3-4\bar\nu)w_{,i}w_{,j}$$

$$+ (4\bar\nu - 1) w_{,k}w_{,k} \delta_{ij}] \qquad (29)$$

where:

$$\int_\Omega \frac{\partial \sigma^*_{ikl}}{\partial x_{0j}} w_{,k}w_{,l} \, d\Omega$$

is a Cauchy-type integral.

Substituting (29) into (28) and taking into account the geometric relationship (3) and the constitutive relationship (4), one obtains the membrane stress expression:

$$\sigma_{ij}(P_0) = \int_\Gamma D_{ijk}p_k \, d\Gamma - \int_\Gamma S_{ijk}u_k \, d\Gamma$$

$$- \frac{1}{2}\int_\Omega T_{ijkl}w_{,k}w_{,l} \, d\Omega$$

$$+ \frac{G}{8(1-\bar\nu)} (2w_{,i}w_{,j} + w_{,k}w_{,k}\delta_{ij}) \ P_0 \in \Omega \ (30)$$

The expressions for D_{ijk} and S_{ijk} are shown in reference 19, p. 130, and T_{ijkl} is of the form:

$$T_{ijkl} = \frac{\phi_{ijkl}(P,P_0)}{r^2} \qquad (31)$$

where:

$$\phi_{ijkl} = -\frac{G}{2\pi(1-\bar\nu)} [8r_{,i}r_{,j}r_{,k}r_{,l} + (1-4\bar\nu)\delta_{ij}\delta_{kl}$$

$$- (1-2\bar\nu)(2\delta_{ij}r_{,k}r_{,l} + 2\delta_{kl}r_{,i}r_{,j} + \delta_{ik}\delta_{jl}$$

$$+ \delta_{il}\delta_{jk}) - 2\bar\nu(\delta_{ik}r_{,j}r_{,l} + \delta_{jk}r_{,i}r_{,l}$$

$$+ \delta_{il}r_{,j}r_{,k} + \delta_{jl}r_{,i}r_{,k})] \qquad (32)$$

therefore, in equation (30) the integral:

$$\int_\Omega T_{ijkl} w_{,k} w_{,l} \, d\Omega = \int_\Omega \frac{\phi_{ijkl}}{r^2} w_{,k} w_{,l} \, d\Omega$$

is a singular integral. Let L be a circle of radius ρ with the centre at point P_0. It can be proved that the function ϕ_{ijkl} satisfies the condition:

$$\int_\Gamma \phi_{ijkl} \, d\Gamma = 0$$

This is the necessary and sufficient condition for the existence of the Cauchy principal value of the aforementioned singular integral.[20]

Thus, the boundary integral equation formulations (17), (19) and (26) for the finite deflection analysis of elastic plates have been obtained. However, these boundary integral equations cannot be solved by the conventional technique due to the nonlinear bending and membrane coupling terms.

Technique of iterative solution

For the purpose of solution, an iterative procedure is developed. In the process of iteration, the values of $w_{,ij}$ and σ_{ij} obtained in the preceding cycle of iteration are substituted into the $q = \bar{q} + h\sigma_{ij} w_{,ij}$ to approximate the pseudo-transverse distributed load for the calculation of the current cycle. Thus, the nonlinear boundary integral equations (17), (19) and (26) are transformed into the linear equations for each cycle of iteration.

After performing the discretization by use of various kinds of boundary elements (e.g. constant element, linear element or high-order element), the boundary integral equations (17) and (19) become a set of algebraic equations:

$$Ax = q(\sigma_{ij}; w_{,ij}) \tag{33}$$

and the equation (26) is transformed into another set of algebraic equations:

$$By = f(w_{,i}) \tag{34}$$

In equations (33) and (34) A and B are the matrices of coefficients of the algebraic equations; x and y are the unknown vectors of boundary variables. On the right-hand side of the equation (33), the vector q is determined by σ_{ij} and $w_{,ij}$, and in (34) the vector f can be found if $w_{,i}$ is known.

The relaxation iterative procedure can be illustrated as follows: suppose that $w^{(K)}$, $\sigma_{ij}^{(K)}$, etc. express the Kth approximations. The initial values of the iteration ($K = 0$) can be set arbitrarily, for example, $\sigma_{ij}^{(0)} = 0$, $\bar{w}_{,ij}^{(0)} = 0$. In the iteration one can solve for $K = 0, 1, 2, 3, \ldots$:

(1) $\quad Ax^{(K+1)} = g[\sigma_{ij}^{(K)}; \bar{w}_{,ij}^{(K)}]$

to obtain $x^{(K+1)}$, and then to evaluate $w^{(K+1)}$, $w_{,i}^{(K+1)}$ and $w_{,ij}^{(K+1)}$ in the domain

(2) $\quad By^{(K+1)} = f[w_{,i}^{(K+1)}]$

to find $y^{(K+1)}$, and then to calculate $u_i^{(K+1)}$ and $\sigma_{ij}^{(K+1)}$.

The iteration is continued until the change in the current estimate of the maximum deflection is small enough, i.e. until:

$$|w_{max}^{(K+1)} - w_{max}^{(K)}| < \epsilon \tag{35}$$

where ϵ is the convergence tolerance; otherwise:

$$\bar{w}_{,ij}^{(K+1)} = \beta w_{,ij}^{(K+1)} + (1 - \beta) \bar{w}_{,ij}^{(K)}$$

where β is the relaxation factor, and then continue the iteration.

It is interesting to note that in (33) and (34) the matrices A, B and other matrices for the calculation of all values in the domain depend only upon the geometric and material parameters of plates. Once these matrices have been formed, they can be stored in the core and used in each cycle of iteration without any change. It can reduce the computing time by about three-quarters.

Numerical examples

Numerical examples are presented to show the feasibility and efficiency of the proposed approach. In these examples constant elements are used. The boundary element division and the interior mesh are illustrated in *Figure 2*.

Example 1: Clamped circular plate under uniform lateral load

Figure 3 shows the numerical results of the maximum deflection w_{max} obtained in this study, which are compared with those given in reference 21. The error of w_{max}/h with 20 boundary elements in comparison with the analytical solutions is less than 0.60%.

Example 2: Clamped square plate under uniform lateral load

In addition to the case of the in-plane immovable boundary condition, i.e. the case of existing constraints on the membrane deformations on the boundary, the case of movable boundary condition (without constraints on the membrane deformations on the boundary) is also considered in this example. Numerical results are shown in *Figure 4*.

Example 3: Simply supported square plate under uniform lateral load

As in Example 2, immovable and movable edges are studied separately. The numerical results obtained and the analytical solutions are shown in *Figure 5* for comparison.
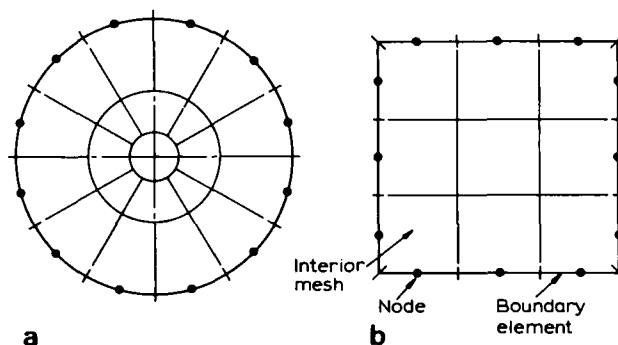

a　　　　　　　　　　**b**

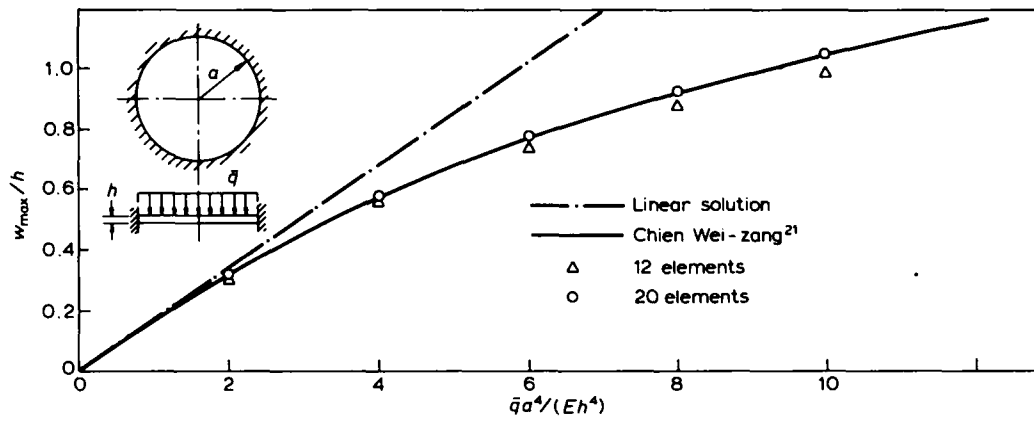*Figure 2* Boundary element division and interior mesh; (a) circular plate, (b) square plate

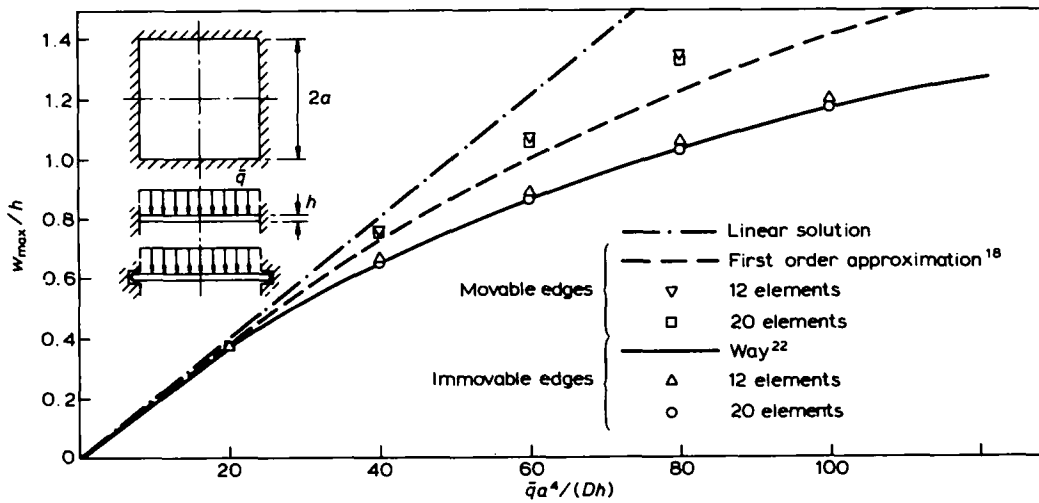*Figure 3* Clamped circular plate ($\nu = 0.3$)

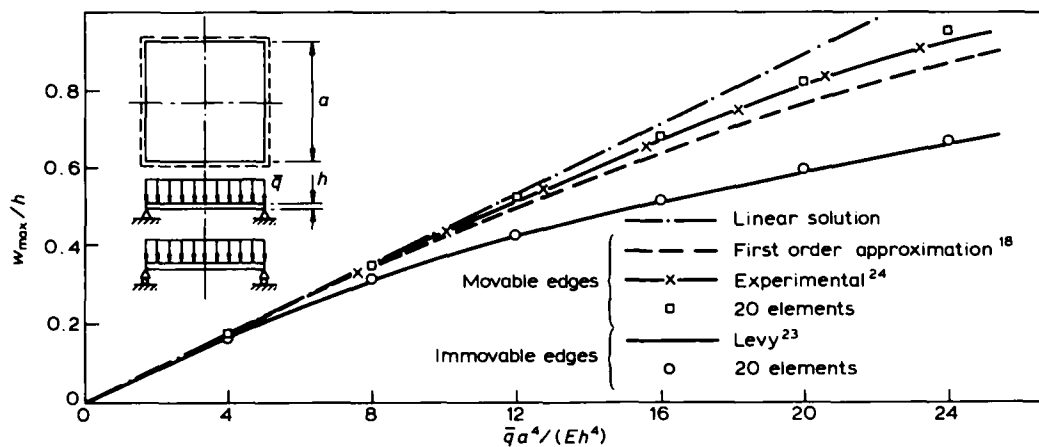*Figure 4* Clamped square plate ($\nu = 0.3$)

Figure 5 Simply supported square plate ($\nu = 0.3$)

In the case of in-plane movable boundary conditions of Examples 2 and 3, the first order approximations given in reference 18 are cited for comparison with the present results for lack of exact solutions. For the simply supported square plate with movable edges, the present results are very close to the experimental data.[24] It appears that the approach developed in this paper gives more accurate results than the first-order approximation.

The rate of convergence depends upon the appropriate choice of relaxation factors. In the above examples, the

cycles of iteration are, on average, less than 20 with an error tolerance $\epsilon = 0.00005$.

Conclusions

The effectiveness of the approach developed in this paper has been shown by numerical examples. This approach appears to be very promising for the finite deflection analysis of thin elastic plates with various boundary conditions and load conditions.

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