Computer Methods in Applied Mechanics and Engineering 96 (1992) 271–287
North-Holland

A weakly singular form of the hypersingular boundary integral equation applied to 3-D acoustic wave problems

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Received 6 February 1991

The composite boundary integral equation (BIE) formulation, using a linear combination of the conventional BIE and the hypersingular BIE, emerges as the most effective and efficient formula for acoustic wave problems in an exterior medium which is free of the well-known fictitious eigenfrequency difficulty. The crucial part in implementing this formulation is dealing with the hypersingular integrals. Various regularization procedures in the literature give rise, in general, to integrals which are still difficult and/or extremely time consuming to evaluate or are limited to the use of special, usually flat, boundary elements. In this paper, a general form of the hypersingular BIE is developed for 3-D acoustic wave problems, which contains at most weakly singular integrals. This weakly singular form can be derived by employing certain integral identities involving the static Green's function. It is shown that the discretization of this weakly singular form of the hypersingular BIE is straightforward and the same collocation procedures and regular quadrature as that used for conventional BIEs are sufficient to compute all the integrals involved. Computing times are only slightly longer than with conventional BIEs. The C^1 smoothness requirement imposed on the density function for existence of the hypersingular BIEs and the possibility of relaxing this requirement are discussed. Three kinds of boundary elements, having different smoothness features, are employed. Numerical results are given for scattering from a rigid sphere at the fictitious frequencies, for values of wavenumber from π to 5π. In essence, with the methodology in this paper the fictitious eigenfrequency difficulty, long associated with the BIE for exterior problems, should no longer be a troublesome issue.

1. Introduction

The solution of the conventional (Helmholtz) BIE formulation for exterior acoustic wave problems is nonunique at the eigenfrequencies of the associated interior problems [1, 2]. In numerical computation, the coefficient matrix of the BIE will yield a large condition number (i.e. be ill-conditioned) when the wavenumber is equal to or near one of the eigenfrequencies. This nonuniqueness is purely a drawback of the mathematical formulation of the problems and does not have any physical significance. Nevertheless, how to circumvent this fictitious eigenfrequency difficulty for the exterior problems has been one of the focal points in the research of BIE applications to acoustic wave problems for a long time.

The CHIEF (Combined Helmholtz Integral Equation Formulation) method, proposed by Schenck [1], is one of the earliest and simplest methods to overcome the nonuniqueness problem. In this method, the system of algebraic equations for the conventional BIE is combined with a few additional equations generated from the Helmholtz integral with the source point in the interior of the closed boundary. A systematic numerical study of the

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CHIEF method was given by Seybert et al. [3], regarding the selection of number and locations of the CHIEF points. Despite its simplicity and success with simple problems, the CHIEF method can be undesirable for practical problems with complicated boundary geometries and wavenumbers in the intermediate range, for which the selection of the CHIEF points can become difficult and trial and error procedures are necessary. Nevertheless, the CHIEF method, especially when used with boundary elements more sophisticated than piecewise constant ones, works well in our experience – better, perhaps, than one would guess from all the attention given to alternatives.

There are indeed many other methods proposed to overcome the fictitious eigenfrequency difficulty, such as the exterior overdetermination method of Piaszczyk and Klosner [4], the modified kernel methods of Ursell [5] and Jones [6]. However, these methods are often difficult to implement or inefficient in computation.

Burton and Miller's formulation [7] (composite BIE), using a linear combination of the conventional (Helmholtz) BIE and the hypersingular (normal derivative) BIE, appears to be the most theoretically desirable approach for dealing with the fictitious eigenfrequency difficulty. The hypersingular BIE alone also fails for exterior problems at another set of eigenfrequencies of the associated interior problem, which is different from that for the conventional BIE. However, Burton and Miller proved that the linear combination of these two BIEs can furnish unique solutions at all frequencies, provided the imaginary part of the coupling coefficient is nonzero.

The most difficult part in implementing this composite formulation is dealing with the hypersingular integrals and intensive work has been done on this aspect for acoustic problems [7–15]. Various regularization procedures presented in these works, in general, give rise to integrals which are still difficult to compute or are limited to the use of special, usually flat, boundary elements. Nevertheless, these works show that the composite BIE is ultimately the most effective and promising method to overcome the fictitious eigenfrequency difficulty. A more detailed review of all these works is provided by Chien et al. [16] in a recent comprehensive work on the composite BIE formulation.

In the present paper, a composite BIE which uses a general form of the so-called hypersingular BIE is developed for 3-D acoustic wave problems. This form contains at most weakly singular integrals and is valid for radiation and scattering problems with arbitrary boundary geometries. Discretization of this weakly singular form of the hypersingular BIE is straightforward. No special numerical integration quadratures are required to compute all the integrals and hence the quadrature for conventional BIE can be applied directly. The computer time to set up the system of equations for this weakly singular form of the hypersingular BIE is found to be only slightly longer than that for the conventional BIE.

Theory [17–20] imposes a C^1 smoothness requirement on the density function of the hypersingular BIEs. There seems to be some question in the literature [16] about the actual need for this requirement depending on the form of the integrals used for computation. Chien et al. [16] are quite explicit that the smoothness requirement for their formulation is only C^0. In their final formulation, the hypersingular integral is transformed to an integral with integrand which is the hypersingular kernel multiplied by the difference of the original density function evaluated at the field point and the source point. This is a special CPV integral and actually demands the same C^1 smoothness for its convergence. This is not recognized in [16]. Indeed, continuity requirements on BIEs are imposed by the order of singularity of the kernels, not by the regularization process used or by the final form of the integrals. Ervin et al. [21] acknowledge the smoothness requirement, but note, as we do later, that good results can be obtained in violation of the C^1 requirement. This is in agreement with the numerical results, if not the reasoning, of Chien et al. [16]. Some discussions on this matter are finally offered here from our understanding of the theory and our own numerical experiments. Our numerical work involves three kinds of boundary elements, namely, C^0 conforming quadratic elements, nonconforming quadratic elements and Overhauser C^1 continuous elements, each having different smoothness features. Examples for scattering from a rigid sphere (of radius a) are presented. The fictitious frequencies examined range from $ka = \pi$ to 5π.

2. Weakly-singular form of the hypersingular BIE

The starting point of the derivation is the following Helmholtz integral:

$$C(P_0)\phi(P_0) = \int_S \left[G(P, P_0) \frac{\partial \phi(P)}{\partial n} - \frac{\partial G(P, P_0)}{\partial n} \phi(P) \right] \mathrm{d}S(P) + \phi^{\mathrm{I}}(P_0) , \tag{1}$$

where ϕ is the total wave (velocity potential or acoustic pressure) satisfying the Helmholtz equation $\nabla^2\phi + k^2\phi = 0$ for time harmonic waves, ϕ^{I} is a prescribed incident wave (for a scattering problem), $G(P, P_0) = \mathrm{e}^{ikr}/4\pi r$ is the full space Green's function for the Helmholtz equation, the coefficient $C(P_0) = 1$, $\frac{1}{2}$ or 0 when the source point P_0 is in the exterior region E, on the boundary S (if it is smooth) or in the interior region B (a body or scatterer), respectively (Fig. 1). Equation (1) with $P_0 \in S$ is the commonly used form of the conventional BIE for acoustic wave problems [22].

The directional derivative of the representation integral ((1) with $P_0 \in E$) at P_0 in direction n_0 is

$$\frac{\partial \phi(P_0)}{\partial n_0} = \int_S \left[\frac{\partial G(P, P_0)}{\partial n_0} \frac{\partial \phi(P)}{\partial n} - \frac{\partial^2 G(P, P_0)}{\partial n \, \partial n_0} \phi(P) \right] \mathrm{d}S(P) + \frac{\partial \phi^{\mathrm{I}}(P_0)}{\partial n_0}$$
$$\forall P_0 \in E . \tag{2}$$

The second integrand is hypersingular in this representation integral when the source point P_0 is on the boundary S. How to handle this hypersingular integral has been the focus of research on applications of the hypersingular BIE. Various regularization methods to reduce the order of singularity can be found in the literature as mentioned in the previous section. Here, a weakly singular form of the hypersingular BIE is derived from the representation integral (2). To do this, it is noticed that the second integral in (2) can be written identically, by subtracting and adding back terms, as follows:

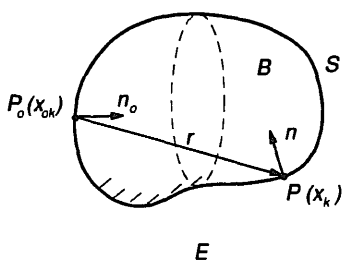

Fig. 1. Notations.

$$
\begin{aligned}
&\int_S \frac{\partial^2 G(P, P_0)}{\partial n\, \partial n_0}\, \phi(P)\, \mathrm{d}S(P) \\
&= \int_S \frac{\partial^2}{\partial n\, \partial n_0}\, [G(P, P_0) - \bar{G}(P, P_0)]\phi(P)\, \mathrm{d}S(P) + \int_S \frac{\partial^2 \bar{G}(P, P_0)}{\partial n\, \partial n_0}\, \phi(P)\, \mathrm{d}S(P) \\
&= \int_S \frac{\partial^2}{\partial n\, \partial n_0}\, [G(P, P_0) - \bar{G}(P, P_0)]\phi(P)\, \mathrm{d}S(P) \\
&\quad + \int_S \frac{\partial^2 \bar{G}(P, P_0)}{\partial n\, \partial n_0}\, [\phi(P) - \phi(P_0) - \phi_{,k}(P_0)(x_k - x_{0k})]\, \mathrm{d}S(P) \\
&\quad + \phi(P_0) \int_S \frac{\partial^2 \bar{G}(P, P_0)}{\partial n\, \partial n_0}\, \mathrm{d}S(P) \\
&\quad + \phi_{,k}(P_0) \int_S \frac{\partial^2 \bar{G}(P, P_0)}{\partial n\, \partial n_0}\, (x_k - x_{0k})\, \mathrm{d}S(P) \quad \forall P_0 \in E\,,
\end{aligned}
\tag{3}
$$

in which $\bar{G}(P, P_0) = \frac{1}{4}\pi r'$ is the Green function in the static case, $(\ )_{,k} = \partial(\ )/\partial x_k$ and summation over k is implied ($k = 1, 2, 3$). By virtue of the identities involving $\bar{G}(P, P_0)$, established in [23], the last two integrals in (3) are found to be

$$
\int_S \frac{\partial^2 \bar{G}(P, P)}{\partial n\, \partial n_0}\, \mathrm{d}S(P) = 0\,, \tag{4}
$$

$$
\int_S \frac{\partial^2 \bar{G}(P, P_0)}{\partial n\, \partial n_0}\, (x_k - x_{0k})\, \mathrm{d}S(P) = \int_S \frac{\partial \bar{G}(P, P_0)}{\partial n_0}\, n_k(P)\, \mathrm{d}S(P)\,, \tag{5}
$$

where the source point P_0 is outside S.

Substituting (4) and (5) into (3) and then (3) into (2), one obtains the following result by arranging the terms:

$$
\begin{aligned}
&\frac{\partial \phi(P_0)}{\partial n_0} + \int_S \frac{\partial^2 \bar{G}(P, P_0)}{\partial n\, \partial n_0}\, [\phi(P) - \phi(P_0) - \phi_{,k}(P_0)(x_k - x_{0k})]\, \mathrm{d}S(P) \\
&\quad + \int_S \frac{\partial^2}{\partial n\, \partial n_0}\, [G(P, P_0) - \bar{G}(P, P_0)]\phi(P)\, \mathrm{d}S(P) \\
&= \int_S \frac{\partial G(P, P_0)}{\partial n_0}\, [\phi_{,k}(P) - \phi_{,k}(P_0)] n_k(P)\, \mathrm{d}S(P) \\
&\quad + \int_S \frac{\partial}{\partial n_0}\, [G(P, P_0) - \bar{G}(P, P_0)][\phi_{,k}(P_0) n_k(P)]\, \mathrm{d}S(P) \\
&\quad + \frac{\partial \phi^{\mathrm{I}}(P_0)}{\partial n_0} \quad \forall P_0 \in S\,,
\end{aligned}
\tag{6}
$$

where the source point P_0 has been placed on the boundary S. It is noticed that in the limit process of $P_0 \to S$, the four integrals in (6) have no jumps if the density function $\phi(P)$ has continuous first derivatives on S. However, if $\phi(P)$ does not have continuous first derivatives on S, it can be shown that a $\log D$ term will appear in the derivation of (6), where D is the distance between P_0 and the point on S to which P_0 is approaching. Thus in the limit as

$P_0 \to S$, an infinite term will be associated with (6) when the density function is not C^1 continuous (cf. [20]).

Equation (6) is the desired weakly singular form of the so-called hypersingular boundary integral equations for the acoustic wave problems. It can also be derived in a classical way similar to that used in [23] for potential problems. When $G(P, P_0) = \bar{G}(P, P_0)$, the weakly singular form of the hypersingular BIE for exterior potential problems [23] is recovered from (6). A 2-D version for interior potential problems was first presented by Rudolphi et al. [24]. A different approach to establish the weakly singular form of the hypersingular BIE for crack scattering problems was developed by Krishnasamy et al. [19] through a novel use of the Stokes theorem instead of the identities. The final BIE of this approach involves some area integrals on the crack surface (an open surface in 3-D) and line integrals along the edge of the crack.

The advantages of (6) are as follows. First, there is no CPV of any kind involved in this formulation. All the four integrals in (6) are at most weakly singular, provided derivatives of the density function $\phi(P)$ are continuous, or more precisely, $\phi(P) \in C^{1,\alpha}$ [19]. This weak singularity will be removed completely after a polar coordinate transformation of the surface element. Second, the discretization of (6) is straightforward. Since regular Gaussian quadrature is sufficient to handle the weakly singular integrals, the numerical schemes [22, 25] employed for the conventional BIE can be applied here for (6) directly. The only special feature of the discretization of (6) is the treatment of the $\phi_{,k}$ term, which is a relatively simple task as will be illustrated in the next section.

However, the important theoretical issue regarding smoothness (alluded to above with (6)) is the source of some confusion in the literature and needs further discussion. It has been shown [19, 20] that for the hypersingular integral

$$\lim_{P_0 \to S} \int_S \frac{\partial^2 G(P, P_0)}{\partial n \, \partial n_0} \phi(P) \, \mathrm{d}S(P) \tag{7}$$

to exist, the derivatives of the density function $\phi(P)$ must be Holder continuous (in the neighborhood of the source point P_0), i.e., $\phi(P) \in C^{1,\alpha}$. One of the terms in the expression of integral (7), which is zero for smooth density functions, can usually be identified as $\log D$ multiplied by, e.g. in the 2-D case, the difference of the slopes of the density function at each side of the point on S to which P_0 is approaching. If the smoothness requirement is not satisfied, this term will go to infinity as P_0 tends to S (i.e. as $D \to 0$) [20]. This smoothness requirement on the density function will exclude, theoretically, the use of the commonly applied C^0 elements in BEM, such as the linear and quadratic elements, in the discretization of (6).

It is of interest to note that in [16] the use of C^0 elements is justified, contrary to the view in the present paper. Moreover, although we cannot justify the use of such elements, we do employ them to see how they work and we find that good data are obtainable with them as do the authors of [16]. Further discussion of this matter is postponed following the numerical examples which employ three kinds of boundary elements, only two of which we can justify on theoretical grounds.

3. Discretization

The discretization of (6) is discussed in this section. First, the $\phi_{,k}$ term in (6) is transformed to a form which involves only the nodal values of the boundary variables ϕ and $\partial\phi/\partial n$ of the problems. Then, the discretized form of the first integral in (6) is presented. Finally, three kinds of boundary elements applied to (6) in the next section are briefly described, namely, C^0 conforming quadratic elements which violate the smoothness requirement, nonconforming quadratic elements which satisfy the smoothness requirement in the neighborhood of the source point, and Overhauser C^1 continuous surface elements which satisfy the smoothness requirement on the entire boundary S.

Let $O\xi_1\xi_2\xi_3$ be a local (curvilinear) coordinate system on a boundary element, where ξ_1 and ξ_2 are along two tangential directions and $\xi_3 = n$ (Fig. 2). One has the following transformation relation:

$$\begin{Bmatrix} \dfrac{\partial\phi}{\partial x_1} \\ \dfrac{\partial\phi}{\partial x_2} \\ \dfrac{\partial\phi}{\partial x_3} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \dfrac{\partial\phi}{\partial \xi_1} \\ \dfrac{\partial\phi}{\partial \xi_2} \\ \dfrac{\partial\phi}{\partial \xi_3} \end{Bmatrix}, \qquad J^{-1} = [\gamma_{kl}] = [\partial\xi_l/\partial x_k], \tag{8}$$

where J^{-1} is the inverse of the Jacobian. Now applying the shape functions on the element, i.e.,

$$\phi = \sum_{\alpha=1}^{K} N_\alpha(\xi)\phi^\alpha, \qquad \frac{\partial\phi}{\partial\xi_1} = \sum_{\alpha=1}^{K} \frac{\partial N_\alpha}{\partial\xi_1}(\xi)\phi^\alpha, \qquad \frac{\partial\phi}{\partial\xi_2} = \sum_{\alpha=1}^{K} \frac{\partial N_\alpha}{\partial\xi_2}(\xi)\phi^\alpha, \tag{9}$$

where $\xi = (\xi_1, \xi_2)$, ϕ^α is the (local) nodal value of ϕ and K is the number of nodes on the element, one obtains the following expression for $\phi_{,k}$ from (8):

$$\begin{Bmatrix} \dfrac{\partial\phi}{\partial x_1} \\ \dfrac{\partial\phi}{\partial x_2} \\ \dfrac{\partial\phi}{\partial x_3} \end{Bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial N_1}{\partial\xi_1} & \dfrac{\partial N_2}{\partial\xi_1} & \cdots & \dfrac{\partial N_K}{\partial\xi_1} & 0 \\ \dfrac{\partial N_1}{\partial\xi_2} & \dfrac{\partial N_2}{\partial\xi_2} & \cdots & \dfrac{\partial N_K}{\partial\xi_2} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{Bmatrix} \phi^1 \\ \phi^2 \\ \vdots \\ \phi^K \\ \dfrac{\partial\phi}{\partial n} \end{Bmatrix}. \tag{10}$$

Let the boundary S be discretized into a total of M surface elements S_m and let N be the total number of nodes on the surface. In the following, the subscripts i and j are reserved to indicate global node numbers, no longer the coordinate directions. Now placing the source point P_0 at node i $(i = 1, 2, \ldots, N)$, one can write the first integral in (6) as follows:

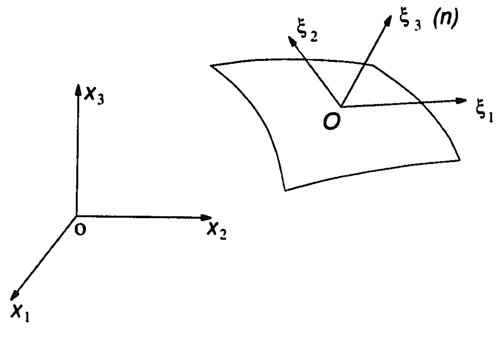

Fig. 2. Coordinate transformation.

$$I = \int_S \frac{\partial^2 \bar{G}(P, P_0)}{\partial n \, \partial n_0} [\phi(P) - \phi(P_0) - \phi_{,k}(P_0)(x_k - x_{0k})] \, \mathrm{d}S(P)$$

$$= \sum_{m=1}^{M} \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n \, \partial n_0} \right]_i [\phi - \phi_i - (\phi_{,k})_i (x_k - x_{ik})] \, \mathrm{d}S(P) , \tag{11}$$

where the subscript i for the kernel function indicates that P_0 is at node i, ϕ_i and $(\phi_{,k})_i$ are the values of the functions at node i and summation over k is implied. The evaluations of the integrals on S_m in (11) are performed in two different ways according to the locations of the source point P_0.

CASE 1: $P_0 \not\in S_m$. In this case the integrals on S_m are regular and one proceeds as follows:

$$I_1 = \sum_m \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n \, \partial n_0} \right]_i [\phi - \phi_i - (\phi_{,k})_i (x_k - x_{ik})] \, \mathrm{d}S(P)$$

$$= \sum_m \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n \, \partial n_0} \right]_i [\phi - \phi_i] \, \mathrm{d}S(P)$$

$$- \sum_m \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n \, \partial n_0} \right]_i [(\phi_{,k})_i (x_k - x_{ik})] \, \mathrm{d}S(P)$$

$$= \sum_m \sum_{\alpha=1}^{K} \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n \, \partial n_0} \right]_i N_\alpha(\xi) |J(\xi)| \, \mathrm{d}\xi [\phi^\alpha - \phi_i]$$

$$- \sum_m \sum_{\alpha=1}^{K} \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n \, \partial n_0} \right]_i N_\alpha(\xi) |J(\xi)| d\xi [x_k^\alpha - x_{ik}] (\phi_{,k})_i$$

$$= \sum_{j=1}^{N} e_{ij}(\phi_j - \phi_i) - \sum_{j=1}^{N} e_{ij}(x_{jk} - x_{ik})(\phi_{,k})_i \quad \text{(sum on global node number)}$$

$$= \begin{bmatrix} e_{i1} & e_{i2} & \cdots & -\sum_{j \neq i}^{N} e_{ij} & \cdots & e_{iN} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_i \\ \vdots \\ \phi_N \end{Bmatrix}$$

$$- [\bar{e}_{i1} \quad \bar{e}_{i2} \quad \bar{e}_{i3}] \begin{Bmatrix} \dfrac{\partial \phi}{\partial x_1} \\ \dfrac{\partial \phi}{\partial x_2} \\ \dfrac{\partial \phi}{\partial x_3} \end{Bmatrix}_i , \tag{12}$$

where the summation over m is carried out on the elements of which the node i is not a local node, e_{ij} and $\bar{e}_{ik}$ are the coefficients composed of the integrals on S_m and the coordinates. It is noticed that the diagonal term e_{ii} is obtained by summing the off-diagonal terms in the same row and need not be determined directly.

CASE 2: $P_0 \in S_m$. In this case the integrals on S_m are weakly singular. In order to keep this weak singularity and hence to ensure the convergence of the numerical integration, the density function in the two-term expansion form must be retained. The transformation relation (8) gives

$$\phi_{,k} = \frac{\partial \phi}{\partial \xi_l} \frac{\partial \xi_l}{\partial x_k} = \frac{\partial \phi}{\partial \xi_1} \gamma_{k1} + \frac{\partial \phi}{\partial \xi_2} \gamma_{k2} + \frac{\partial \phi}{\partial n} \gamma_{k3} .$$

Applying this and the shape functions in (9), one can write

$$\begin{aligned} I_2 &= \sum_m \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n\, \partial n_0}\right]_i [\phi - \phi_i - (\phi_{,k})_i (x_k - x_{ik})]\, \mathrm{d}S(P) \\ &= \sum_m \sum_{\alpha=1}^{K} \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n\, \partial n_0}\right]_i \Big\{ N_\alpha(\xi) - N_\alpha(\xi_i) \\ &\qquad - \left[\gamma_{k1} \frac{\partial N_\alpha}{\partial \xi_1}(\xi_i) + \gamma_{k2} \frac{\partial N_\alpha}{\partial \xi_2}(\xi_i)\right](x_k - x_{ik}) \Big\} |J(\xi)|\, \mathrm{d}\xi\, \phi^\alpha \\ &\quad - \sum_m \int_{S_m} \left[\frac{\partial^2 \bar{G}}{\partial n\, \partial n_0}\right]_i [\gamma_{k3}(x_k - x_{0k})] |J(\xi)|\, \mathrm{d}\xi \left(\frac{\partial \phi}{\partial n}\right)_i , \end{aligned} \tag{13}$$

where the summation over m is carried out on the elements surrounding the node i. It is observed that the weakly singular feature of the original integral is retained in the above discretization, as desired.

Thus, the integral in (11) is obtained by $I = I_1 + I_2$, which is the discretized form of the first integral in (6). The discretizations of the other three integrals in (6) are much easier since the distinction between $P_0 \not\in S_m$ and $P_0 \in S_m$ is unnecessary and all the integrals involved are at most weakly singular. The weak singularities in all the cases can be removed after a polar coordinate transformation [25] and regular Gaussian quadrature is sufficient to be employed in the computation of all the integrals.

In this paper, three kinds of boundary elements are applied in the discretization of (6) (and also that of the conventional BIE in the composite formulation). First, conforming quadratic elements (Fig. 3) are tested, which belong to the C^0 class and are widely used in the BEM literature [3, 13, 16, 22, 25]. The smoothness requirement imposed on (6) is violated by these elements. Nevertheless, as mentioned in the previous section, if smoothness requirements are relaxed somewhat, acceptable results can be often achieved by these elements. The major practical problem with these elements is the nonuniqueness of the derivatives of the density function at the nodal points. For example, in the computation of the integrals in I_2 (eq. (13)), different values of the derivatives of ϕ and the normal of the discretized surface at the source point P_0 are used corresponding to different elements surrounding P_0, in order to achieve the convergence of these integrals. In I_1 (eq. (12)), the derivatives of ϕ at P_0, which are

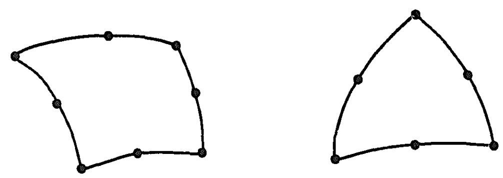

Fig. 3. Conforming quadratic elements.

determined by expression (10), can assume different values when different elements are used in the evaluation of (10). A reasonable strategy then is to use the averaged values for these derivatives as is adopted in this work. Second, the nonconforming quadratic elements [19] (Fig. 4) are applied, where the nodes are moved inside the elements so that the smoothness requirement can be met in the neighborhood of every node. It is relatively easier to implement these elements in the computer codes since no averaging process is needed. Finally, the Overhauser C^1 continuous surface elements developed by Hall and Hibbs [26–28] (Fig. 5) are used, which meet the smoothness requirement on the entire boundary. For the quadrilateral element, four nodes are placed at the corners. Twelve auxiliary nodes on the surrounding elements are also used in the definition of the shape functions (16 altogether). For the triangular element, there are three nodes on the element and nine auxiliary nodes (12 shape functions altogether). The Overhauser elements are expected to give better numerical results than the quadratic elements since they provide a smooth representation of both the geometry of the surface and the density function. Another important feature of the Overhauser elements is that for a fixed number (M) of boundary elements, they produce a much smaller system of linear algebraic equations (number of nodes N is approximately equal to M, assuming most of the elements used are quadrilateral), compared with the conforming quadratic elements ($N \approx 3M$) and the nonconforming quadratic elements ($N \approx 8M$). The comparisons of the numerical results obtained by the above three kinds of elements are presented in the next section.

The discretized equation of the weakly singular form of the hypersingular BIE, eq. (6), is finally written as

$$\boldsymbol{A}_{\mathrm{h}}\boldsymbol{x} = \boldsymbol{b}_{\mathrm{h}} , \tag{14}$$

where $\boldsymbol{A}_{\mathrm{h}}$ is the N by N matrix of the coefficients, $\boldsymbol{x}$ the unknown vector and $\boldsymbol{b}_{\mathrm{h}}$ the known vector composed of the values of the incident wave (for scattering problem) or the prescribed boundary conditions (for radiation problem), at the nodes. Suppose that the discretized equation of the conventional BIE, eq. (1), with $P_0 \in S$, is given by

$$\boldsymbol{A}_{\mathrm{c}}\boldsymbol{x} = \boldsymbol{b}_{\mathrm{c}} , \tag{15}$$

then the composite formulation which is free of any fictitious eigenfrequencies can be represented by

$$(\boldsymbol{A}_{\mathrm{c}} + \beta \boldsymbol{A}_{\mathrm{h}})\boldsymbol{x} = (\boldsymbol{b}_{\mathrm{c}} + \beta \boldsymbol{b}_{\mathrm{h}}) , \tag{16}$$

where β is the coupling *complex* constant ($\mathrm{Im}(\beta) \neq 0$).

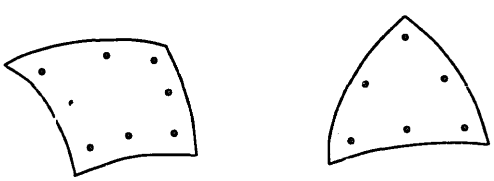

Fig. 4. Nonconforming quadratic elements.

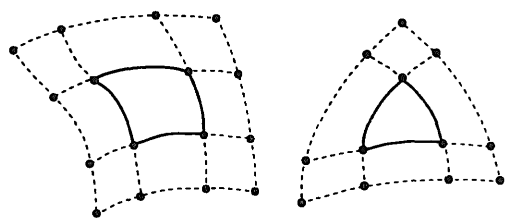

Fig. 5. Overhauser C^1 continuous elements.

4. Numerical examples

The numerical examples are for the scattering problem of a plane incident wave ϕ^I from a rigid sphere ($\partial\phi/\partial n = 0$ on the boundary) (Fig. 6). The magnitudes of the ratio of the scattering wave ϕ^S to ϕ^I at $r = 5a$ are plotted versus the angle θ for various fictitious frequencies of this problem, ranging from $ka = \pi$ to 5π, and compared with the analytical solution [22, 29]. In all the cases, M is the total number of elements on the *whole* spere and N the total number of nodes.

Figure 7 shows the results by the three BIE formulations with Overhauser elements, for $ka = 4.4934$, which is the second fictitious frequency of the conventional BIE. The convention-

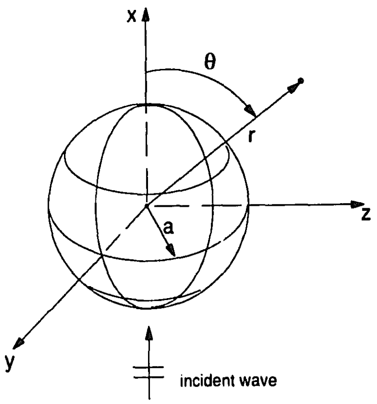

Fig. 6. The sphere.

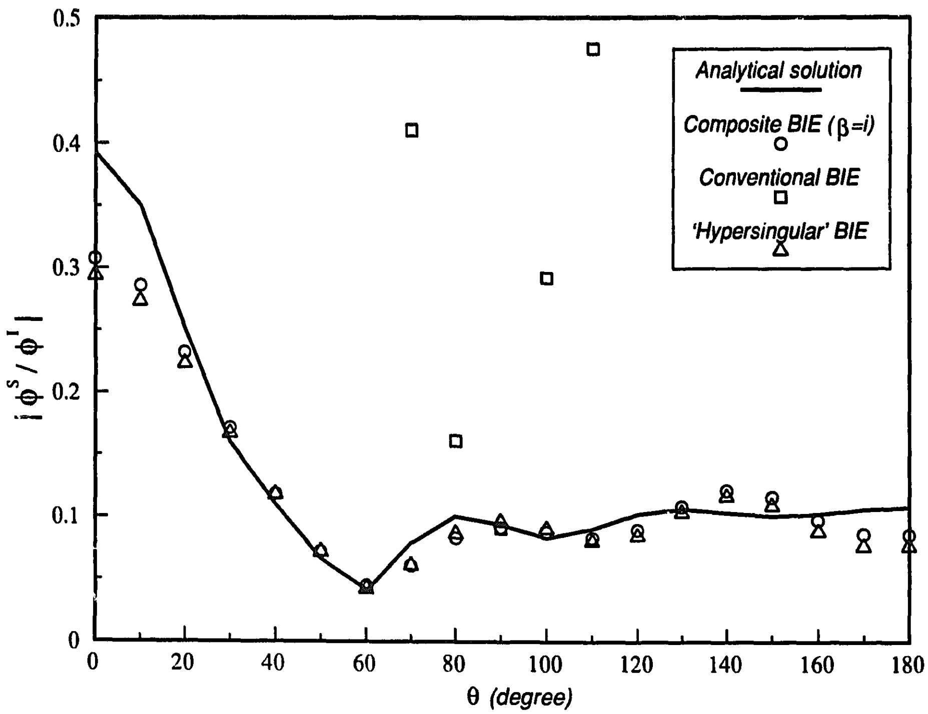

Fig. 7. Scattering wave for $ka = 4.4934$, Overhauser elements ($M = 56$, $N = 54$).

al BIE, eq. (15), breaks down at this frequency as shown in the figure, which is also indicated clearly by a large condition number of the coefficient matrix. The hypersingular BIE, eq. (14), and the composite BIE, eq. (16), on the other hand, give fairly good results with a relatively small number of elements.

Figure 8 presents the results of a test on the choice of the coupling coefficient β used in the composite formulation, eq. (16), for $ka = \pi$. The hypersingular BIE corresponds to the case when β tends to infinity (along the imaginary axis). Best results with 80 Overhauser elements are obtained when $\beta = 0.3333\text{i}$. This is in agreement with the conclusion, first made by Meyer et al., based also on numerical experiments [9], that the best performance of the composite BIE formulation is achieved when the coupling coefficient β is related to the wave number by $\beta = \text{i}/k$.

Figure 9 is a comparison of the three types of boundary elements applied in this paper, for $ka = 2\pi$, namely C^0 conforming, nonconforming and Overhauser elements. First, the results by the three elements are compared for the same number of elements ($M = 56$). The nonconforming elements with $N = 432$ (square syr ols) provide very accurate results over the whole range of values of θ except near the backscattering direction, while the C^0 conforming elements with $N = 154$ (shallow triangles) give fairly good results for most values of θ except near the forward scattering direction. However, for $M = 56$, the results by Overhauser elements with $N = 54$ (solid triangles) are not acceptable. It seems that the small number of Overhauser elements used cannot represent the variations of the boundary variables very well at this frequency. (Note that the same number of Overhauser elements gives fairly good results for $ka = 4.4934$, Fig. 7.) Second, to compare the performance of the three elements for a fixed number of nodes used, the number of C^0 conforming elements is increased to $M = 153$ with $N = 427$, which is almost as many as that of the nonconforming elements. The accuracy of the results (asterisks) are improved to a level close to those of the nonconforming elements (squares) for most of values of θ. Because the effort to obtain a mesh of Overhauser elements

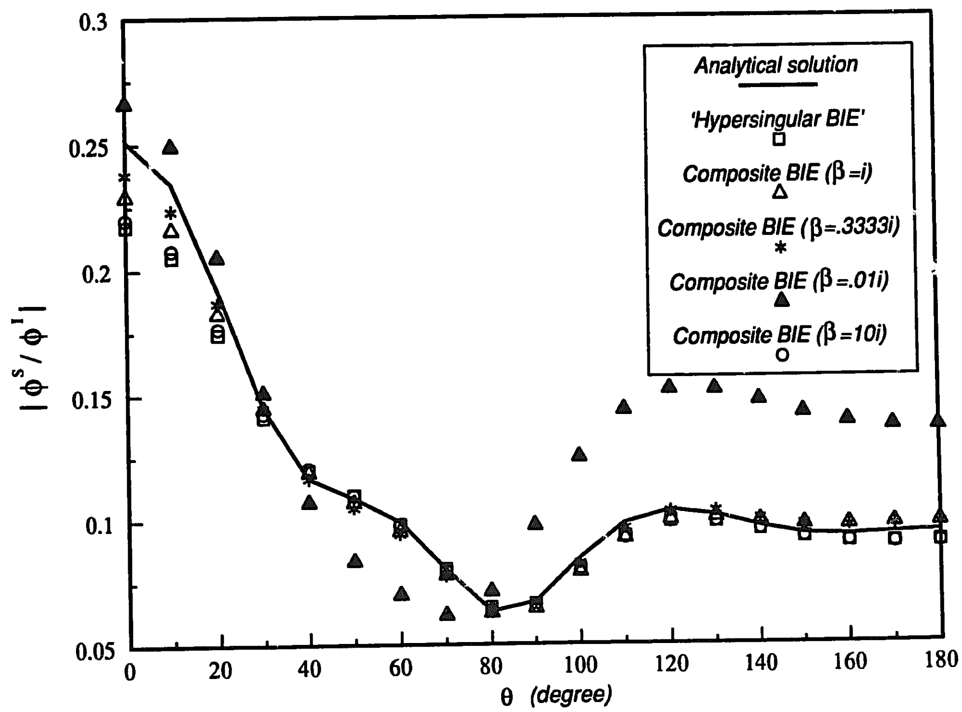

Fig. 8. Test on the coupling coefficient β, $ka = \pi$, Overhauser elements ($M = 80$, $N = 78$).

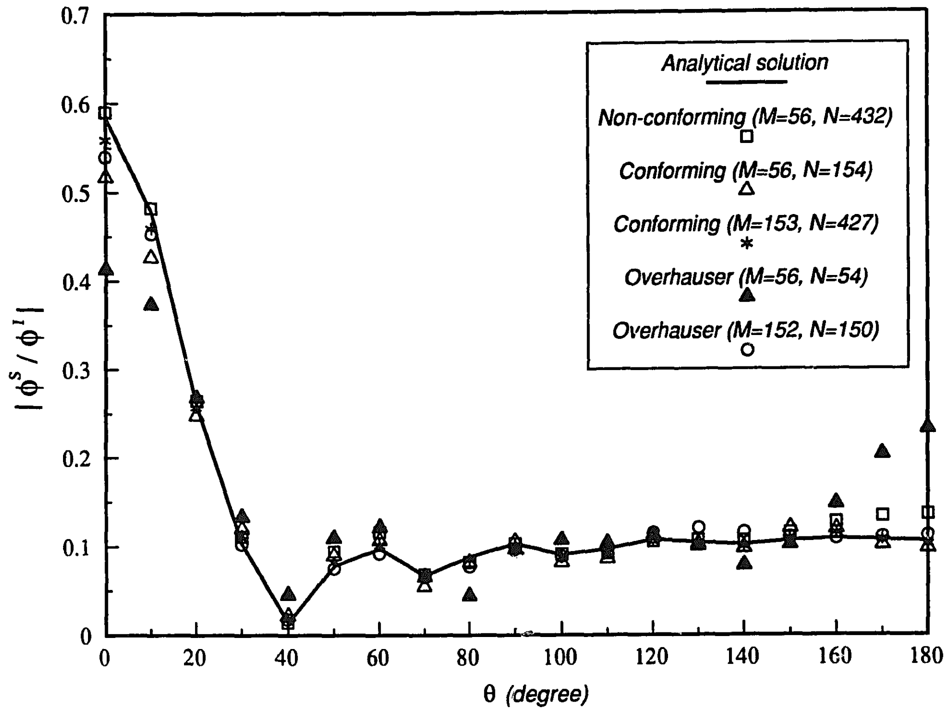

Fig. 9. Comparison of the elements, $ka = 2\pi$, composite BIE ($\beta = 0.15i$).

sufficiently fine to yield about 400 nodes would be prohibitive with existing software available to the authors, a finer mesh of Overhauser elements with $M = 152$ and $N = 150$ is used. The results (circles) are improved greatly although the number of nodes used is only one-third of that of the nonconforming elements. A more detailed study of Overhauser elements compared

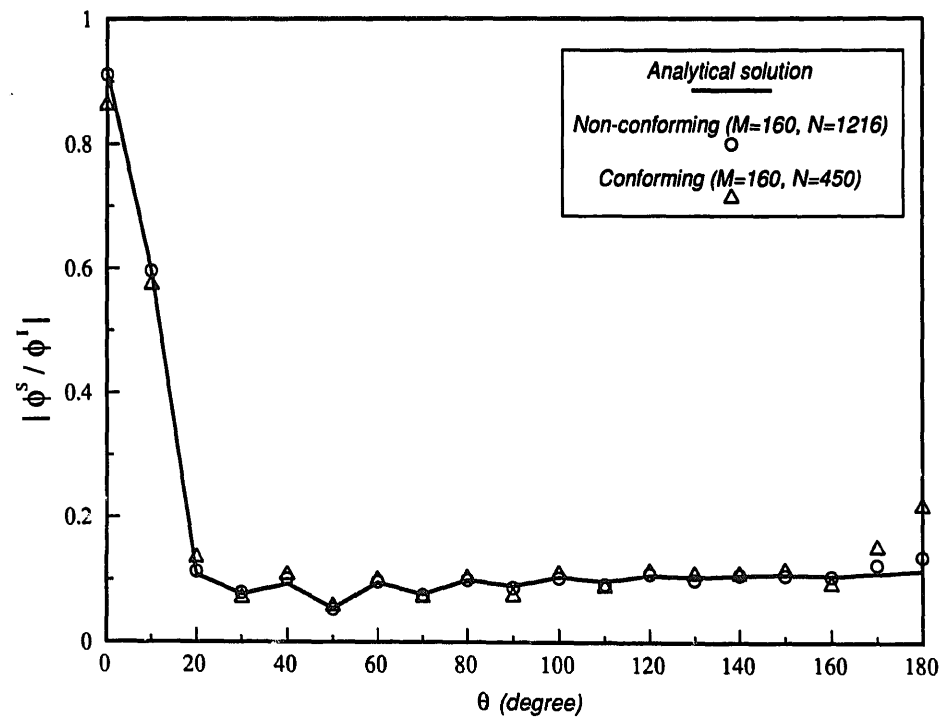

Fig. 10. Result for $ka = 3\pi$, composite BIE ($\beta = 0.1i$).

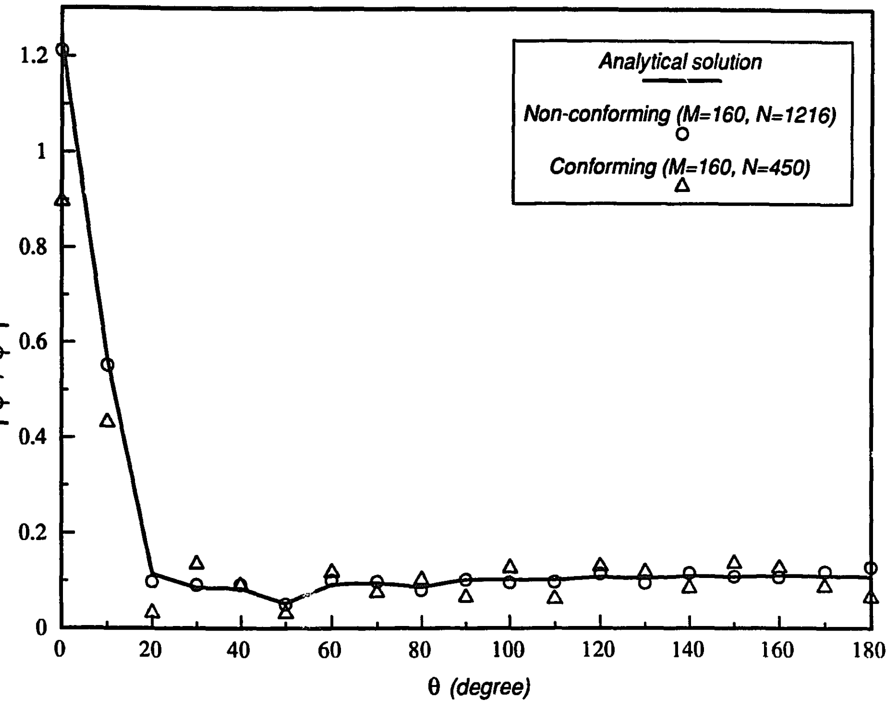

Fig. 11. Result for $ka = 4\pi$, composite BIE ($\beta = 0.08i$).

with C^0 conforming and nonconforming elements, regarding efficiency and accuracy, will be presented in a future paper.

Figures 10, 11 and 12 are the results for $ka = 3\pi, 4\pi$ and 5π, respectively. The results clearly demonstrate the effectiveness of the composite BIE formulation with the weakly singular form of the hypersingular BIE as the key ingredient.

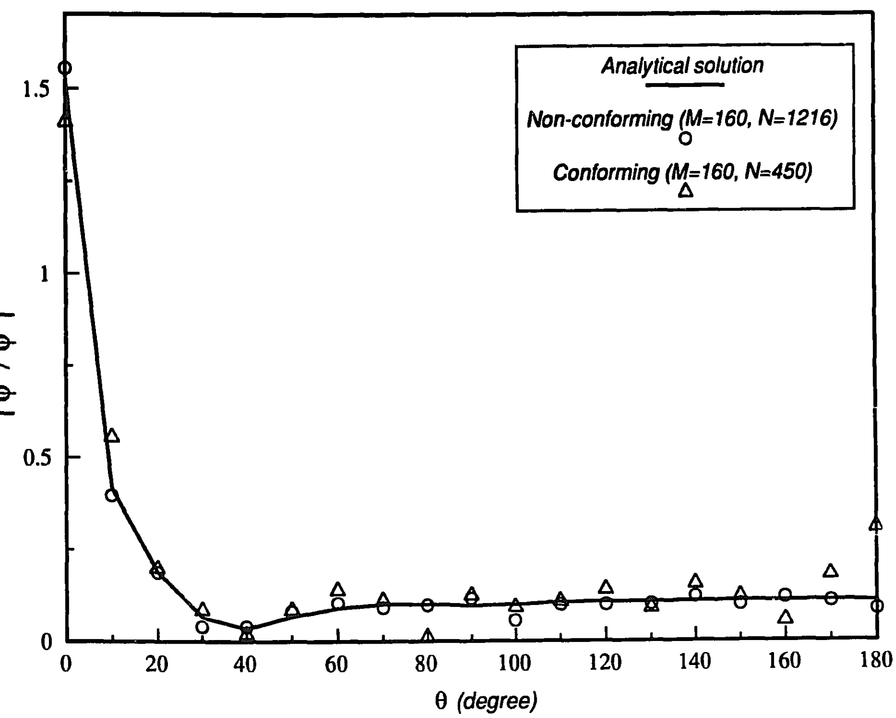

Fig. 12. Result for $ka = 5\pi$, composite BIE ($\beta = 0.06i$).

It should also be pointed out that $ka = 5\pi$ is an uncommonly high frequency for a successful BIE solution for a 3-D scattering problem. Moreover BIE results at such high frequencies are probably more than needed in most problems of this type since high frequency approximation methods can be employed above ka of about 10. Thus the numerical results show that the composite BIE formulation developed in this paper is capable of providing a suitable bridge between low and high frequency approximations.

5. Discussion

This paper is a continued effort of the previous work [24, 17–20, 23] on applications of hypersingular BIEs. It is understood now that singular or hypersingular BIEs in applications of boundary element methods can, in fact, be written in weakly singular forms. Weakly singular forms of various BIEs can be derived from the representation integrals directly, if certain properties of the fundamental solutions are recognized and utilized [23] as has been done for (6), or equivalently, certain analytical manipulations are applied [19, 24].

The weakly singular form of the hypersingular BIE has many advantages. Since the same numerical integration scheme as that used for conventional BIE can be applied to compute all the integrals in this form of the hypersingular BIE, the discretization of the latter is straightforward. It is also found that the CPU time consumed to generate the matrix for the hypersingular BIE, eq. (14), is only about 1.2 times the CPU time for the conventional BIE, eq. (15). This efficiency in the formation of the system is in strong contrast with that of the double integral (operator) approach to the hypersingular BIE (or Galerkin approach), introduced by Burton and Miller [7] and studied numerically in [13, 30, 31]. In [30, 31] it is reported that the formation time for the composite BIE with the double integral approach to the hypersingular BIE is considerably higher than that for the regular BIE by a factor of more than 20.

The efficiency of the present work is also contributed by the integration scheme developed in [32]. In this integration scheme, the number of the Gauss points is not fixed for all the elements, but varies according to the distance and orientation of the source point to the element, the size of the element and the wavenumber. It should be noted that the composite BIE, eq. (16), needs to be applied only when needed, i.e. only when the conventional BIE, eq. (15), fails (indicated by a large condition number of the matrix), although it is understood that the fictitious eigenfrequencies are more closely spaced with increasing frequency.

Regarding the smoothness issue and the type of elements which are acceptable for use with hypersingular integral, however they may be written, e.g. in less (weakly) singular forms, we offer the following remarks. If the density function $\phi(P)$ multiplying the hypersingular kernel in (7) is represented by shape functions which are only C^0, or more precisely $C^{0,\alpha}$, at a collocation point P_0, the limit in (7) does not exist because of the presence of an unbounded term added to other finite terms. If this unbounded term (which is not present for C^1, or more precisely $C^{1,\alpha}$, shape functions) is ignored, and computation made with the finite terms only, it appears, nevertheless, that good results can be obtained as discussed above, and in [20, 21]. The comments and numerical experiments in [20] suggest caution and concern about results which can be scale dependent, and we are uneasy at best about trusting any data or building a computational scheme in flagrant violation of theoretical demands. Nevertheless, data can be good, albeit surprising to us.

The situation in [16] is noteworthy in that the authors insist that the use of C^0 elements is justified with their formulation. The key integral in the final hypersingular BIE formulation presented as (40) or (43) in [16] is

$$\int_{\Delta S} [\phi(P) - \phi(P_0)] \frac{\partial^2 \bar{G}(P, P_0)}{\partial n \, \partial n_0} \, \mathrm{d}S(P) ,$$

where ΔS is the area containing the elements surrounding the source point P_0. This is a special kind of CPV integral, which is similar to the integral in [8] (the dynamic kernel is replaced by the static one), and demands special treatment. It is true that the existence of this integral 'involves a more stringent requirement on the density' than that of the ordinary CPV integrals, as stated in [16]. A necessary condition for the existence of this integral, or equivalently that of integral (7), is the so-called Lyapunov condition [33] which reads (assuming certain smoothness of the surface)

$$\left| \int_0^{2\pi} [\phi(P) - \phi(P_0)] \, \mathrm{d}\theta \right| < K r^{1+\rho} , \quad K, \rho > 0 ,$$

where (r, θ) are the coordinates of P in the polar coordinate system with origin at P_0. This Lyapunov condition for surface integrals can be regarded as a weaker version (in an averaging sense) of the $C^{1,\alpha}$ requirement for line integrals. However, it can easily be shown that C^0 quadratic elements *do not* satisfy the Lyapunov condition at the nodes, by considering, e.g., two adjacent quadrilateral elements with source point P_0 being the middle node on the common edge. This is in conflict with the statement that such elements are justified by the Lyapunov condition as made in [16] (eq. (41)). Thus the authors' use of C^0 (conforming) quadratic elements in [16] for their integrals appears not to be justified either. Most importantly, continuity requirements on ϕ are not greater for the present method or those of [19] because of the *explicit* appearance of $\phi_{,k}$ in these works. The continuity requirements are imposed by the order of singularity of the kernels, not by the regularization process used; the latter may or may not involve a derivative like $\phi_{,k}$ explicitly. Indeed, it should be noted that in the last step of discretization in [16], i.e. eq. (48), the constants α_i and β_i are actually the values of the derivatives of the shape functions at the source point and hence the $\phi_{,k}$ terms are *implicitly* subtracted from the density function in order to reduce the order of singularity. This procedure has the same effect as the method of this paper. The difference is that the subtraction of the $\phi_{,k}$ terms is done in the analytical formulation stage in this paper as opposed to the discretization stage in [16].

All of the considerations above suggest that the present formulation for exterior acoustics problems is a highly efficient and competitive one. Indeed, on the competitive issue, the recent work [31, 34] with domain-based formulations for exterior problems, based on the work of Givoli and Keller [35, 36], seems to be motivated at least in part by the following assumption: the most analytically complete BIE formulation, due to Burton and Miller [7], is entirely free of the fictitious eigenfrequency difficulty but the formulation presents serious computational drawbacks or is otherwise unattractive because of the presence of hypersingular integrals. However, there exists a body of fairly recent literature on hypersingular integral equations, e.g. [10, 16, 17, 19, 21, 23, 37–46], which suggests a variety of modern treatments of hypersingular integrals, and which breathes new life into the Burton–Miller formulation. Admittedly, most of the hypersingular references mentioned focus on elastodynamics or the mathematics involved without specification application, rather than acoustics, but the ideas in these papers are entirely relevant to the issue at hand. Thus, the Burton–Miller formulation is becoming quite attractive as work with hypersingular integrals progresses. We hope that the present work is an illustration of this.

It is believed that the weakly singular forms of the hypersingular BIEs will find more and more applications in problems for which the conventional BIEs are inefficient or fail. One such application to *elastodynamic* problems is already underway.

Acknowledgement

Partial support for this work was provided by the US Office of Naval Research, Applied Mechanics Division, under Contract N00014-89-K-0109, Yapa Rajapakse scientific officer, and by the National Science Foundation under Grant NSF MSS-8918005. Thanks are due many colleagues, especially Dr. Guna Krishnasamy for many helpful conversations on hypersingular integrals, and Dr. W.S. Hall for his kindness in providing crucial materials on the Overhauser elements.

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