Some identities for fundamental solutions and their applications to weakly-singular boundary element formulations

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Some integral identities for the fundamental solutions of potential and elastostatic problems are established in this paper. With these identities it is shown that the conventional boundary integral equation (BIE), which is generally expressed in terms of singular integrals in the sense of the Cauchy principal value (CPV), and the derivative BIE, which is similarly expressed in terms of hypersingular integrals in the sense of the Hadamard finite-part (HFP), can both be written as weakly-singular integral equations in a systematic approach. Discretization of the weakly-singular BIE leads to the weakly-singular boundary element formulation equivalent to the method of using the rigid body displacement to determine the diagonal submatrices, which involve the CPV terms and the geometric matrix C, in the conventional BEM. The discretization of the weakly-singular derivative BIE possesses a similar feature, i.e. no CPV and HFP are involved. All these suggest that the practice of calculating CPV or HFP (for boundary integrals) and the geometric matrix C, either analytically or numerically, is *unnecessary* in the BEM. The approach developed in this paper is applicable to other problems such as plate bending, acoustics and elastodynamics.

Key Words: identities for fundamental solutions, weakly-singular BIE and BE formulation, rigid body displacement method, potential and elastostatic problems.

INTRODUCTION

A universal practice in the BIE literature is to express the equations in terms of CPV integrals. The CPV terms are a consequence of analytically extracting the principle part, or free term coefficient, from the total integral. There are several ways to deal with these singular integrals. One choice is to evaluate the CPV analytically. This can be done for two-dimensional problems with primitive elements (e.g. constant or linear elements) but is limited to simple elements due to the complexity of performing the integrations in closed form. A second choice is to evaluate the CPV numerically. A number of quadrature formulas have been developed, but due to the singular integrands, numerical computation of the CPV's is often inaccurate or requires great computation effort, which inevitably reduces both the accuracy and efficiency. Research continues in this regard[1]. A third choice is to avoid the direct computation of the CPV. A simple way to do so is through the use of rigid body displacement method[2,3] in which the diagonal submatrices containing the CPV and the principal part, or geometric (coefficient) matrix **C** (which is $\frac{1}{2}$**I** for a smooth boundary), are determined by the off-diagonal submatrices.

A question concerning this method has been: Is this an *ad hoc* approach or a general procedure? Although numerical examples of potential and elastostatic problems have shown that the relationship underlying the rigid body displacement method exists even when the diagonal submatrices (containing CPV and the **C** matrix) are determined analytically, there is apparently no firm mathematical basis to answer the above question. This may be why the first two techniques (evaluate the CPV analytically or numerically) remain active research topics and are commonly used.

As early as in 1977, it was shown by Rizzo and Shippy[4] that the well-known BIE for elastostatic problems can be written in a weakly-singular form by using an integral expression for the geometric matrix **C** obtained by the rigid body translation solutions onto the BIE. This result reveals that the BIE is, in fact, weakly-singular, contrary to the widely held conception that the BIE is singular in the sense of CPV. Weakly-singular forms of the BIE for elastodynamic problems are also possible[5], even though the rigid body motion solution is not directly applicable. Unfortunately, it seems that the significance of the weakly-singular BIE's has not been fully appreciated in the BEM community. The singular integrals in the

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conventional BIE's have been considered the central issue in BEM work for the last decade, while the rigid body displacement idea has been considered *ad hoc* and often only used as a way of checking the accuracy of the diagonal submatrices when they are calculated by direct approaches.

Similarly to the conventional BIE, the derivative BIE has generally been written in terms of Hadamard finite-part integrals (HFP), which have stronger singularities and are divergent in the sense of CPV. The derivative BIE is obtained by taking the spatial derivative (usually normal derivative) of the BIE at the source point and is extremely useful for some problems, e.g. the crack problems and plate bending problem in which the BIE alone cannot provide enough algebraic equations. Numerical treatment of the HFP integrals is more troublesome than the CPV integrals. A common practice is to regularize the derivative BIE through integration by parts to reduce the stronger singularity. Alternatively, Rudolphi, *et al.*[6] have shown that the derivative BIE for 2D potential problems can be written in a weakly-singular (or regular) form if certain analytical manipulations are performed in its derivation. Similar weakly-singular derivative BIE's for the scattering problems by a crack (an open surface) in 3D acoustic and elastic media are presented in Ref. 7.

In this paper, weakly-singular BIE's and weakly-singular derivative BIE's for both potential and elastostatic problems (2D and 3D with closed and infinite domains) are derived in a straightforward and systematic way by applying three readily established identities for the fundamental solution of each problem. It is shown that the boundary element discretization of the weakly-singular BIE's (independent of the types of boundary elements) gives rise to a weakly-singular boundary element formulation, in which no calculations (either analytical or numerical) of the CPV's and the **C** matrices are needed. This result is exactly the same as that by the rigid displacement method when applied to the discretized equations of the conventional singular BIE. Thus it is shown that there is an explicitly mathematical justification for the rigid body displacement method. Furthermore, the discretized weakly-singular derivative BIE's, not shown in this paper, will not require the evaluation of either the HFP's or CPV's.

The mathematical approach developed in this paper to establish the weakly-singular BIE and the weakly-singular derivative BIE is readily applicable to other problems such as plate bending, acoustics and elastodynamics.

SOME IDENTITIES FOR THE FUNDAMENTAL SOLUTIONS

Some properties of the fundamental solutions for potential and elastostatic problems are established in this section. Three integral identities satisfied by each fundamental solution are presented and physical interpretations are provided. These identities are subsequently applied in derivations of weakly-singular BIE's and derivative BIE's in the next two sections.

Let $V \subset \mathbb{R}^m$ be an *arbitrary closed* (interior) domain ($m = 2$ or 3 for 2D or 3D problems, respectively), $S = \partial V$ and $E = \mathbb{R}^m - (V \cup S)$ as shown in Fig. 1. The following sifting properties of the Dirac-delta function[8] $\delta(P, P_0)$ will be applied in the establishment of the various identities:

$$\int_V f(P)\,\delta(P, P_0)\,dV(P) = \begin{cases} f(P_0), & \forall P_0 \in V, \\ 0, & \forall P_0 \in E; \end{cases} \tag{1}$$

and

$$\int_V f(P)\frac{\partial}{\partial x_{0j}}\,\delta(P, P_0)\,dV(P) = \begin{cases} -\dfrac{\partial}{\partial x_{0j}} f(P_0), & \forall P_0 \in V, \\ 0, & \forall P_0 \in E; \end{cases} \tag{2}$$

where $f(P)$ is an arbitrary continuous function in $\mathbb{R}^m$, x_{0j} and x_j are the coordinates of the source point P_0 and the field point P, respectively.

(A) Potential problems

The fundamental solution $u^*(P, P_0)$ of the potential problem is defined by the following (operational) equation,

$$\nabla^2 u^*(P, P_0) + \delta(P, P_0) = 0, \quad \forall P, P_0 \in \mathbb{R}^m, \tag{3}$$

where $\nabla^2(\ ) = \partial^2(\ )/\partial x_k \partial x_k = (\ )_{,kk}$ (index notation is used in this paper). We can show that the following three identities are satisfied by $u^*(P, P_0)$:

The first identity

$$\int_S \frac{\partial}{\partial n}\, u^*(P, P_0)\, dS(P) = \begin{cases} -1, & \forall P_0 \in V, \\ 0, & \forall P_0 \in E; \end{cases} \tag{4}$$

The second identity

$$\int_S \frac{\partial^2}{\partial n \partial n_0}\, u^*(P, P_0)\, dS(P) = 0, \quad \forall P_0 \in V \cup E; \tag{5}$$

The third identity

$$\int_S n_k(P)\frac{\partial}{\partial n_0}\, u^*(P, P_0)\, dS(P) - \int_S (x_k - x_{0k})\frac{\partial^2}{\partial n\,\partial n_0}\, u^*(P, P_0)\, dS(P) = \begin{cases} n_{0k}(P_0), & \forall P_0 \in V, \\ 0, & \forall P_0 \in E; \end{cases} \tag{6}$$

where $n = n(P)$ is the outward normal at $P \in S$ with direction cosines $n_k(P)$ and $n_0 = n_0(P_0)$ indicates an arbitrary direction with direction cosines $n_{0k}(P_0)$ as depicted in Fig. 1. It is emphasized that the surface S in the above identities is an arbitrary closed surface in $\mathbb{R}^m$. There are several ways to establish the identities (4–6) and we present two approaches in this section.

An operational approach:

The delta function $\delta(P,P_0)$ is a generalized function in the sense that it must be understood by its actions (or operations) on other functions as shown in expressions (1–2). To satisfy equation (3), we recognize that the fundamental solution $u^*(P, P_0)$ must also be considered a generalized function. For such generalized functions, however, the operation of integration by parts, which is the essence of the Gauss' theorem (or the Green's theorem for 2D problems), is still valid[8]. Thus, we can apply these theorems to the fundamental solution $u^*(P, P_0)$ regardless of its singularity at the source point P_0 in the sense of

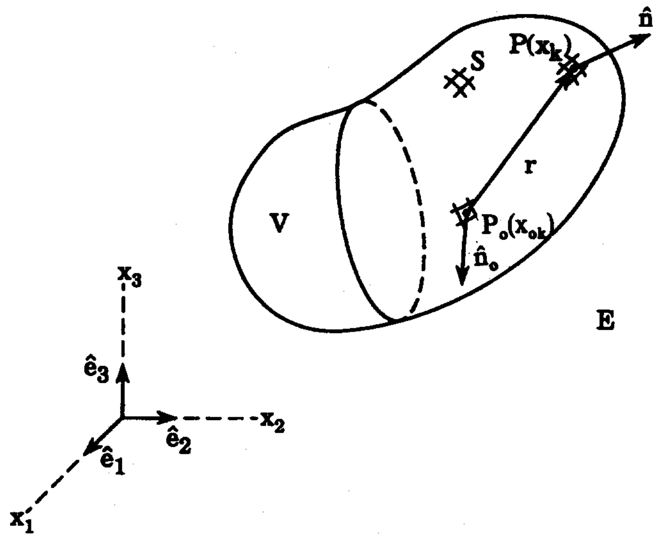

Fig. 1. An arbitrary closed domain V in $\mathbb{R}^m$

ordinary functions. We refer to this an *operational approach*. It has been loosely employed in the BEM literature (e.g. Refs 2 and 9) for some time, but the above arguments have not been mentioned and the generalized nature of the treatment of the functions has not been recognized.

To establish the first identity, we start with equation (3) and integrate both sides over the domain V to obtain

$$\int_V \nabla^2 u^*(P, P_0) dV(P) + \int_V \delta(P, P_0) dV(P) = 0. \quad (7)$$

Then by Gauss' theorem (in a generalized sense), the first integral is

$$\int_V \nabla^2 u^*(P, P_0) dV(P) = \int_V u^*_{,kk}\, dV = \int_S u^*_{,k} n_k\, dS$$

$$= \int_S \frac{\partial}{\partial n} u^*(P, P_0)\, dS(P)$$

and by substituting this result into equation (7) and using expression (1) for the second integral (where $f(P) = 1$), one immediately obtains the first identity (4).

The derivative of $u^*(P, P_0)$ with respect to $n_0(P_0)$ must also satisfy the differential equation

$$\nabla^2\left[\frac{\partial}{\partial n_0} u^*(P, P_0)\right] + \frac{\partial}{\partial n_0}\delta(P, P_0) = 0, \quad \forall P, P_0 \in \mathbb{R}^m. \quad (8)$$

Integrating both sides of this equation over the domain V, as we did for equation (3), applying Gauss' theorem to the first integral and expression (2) to the second integral (where $f(P) = 1$ and $\partial(\ )/\partial n_0 = n_{0j}\partial(\ )/\partial x_{0j}$), one obtains the second identity (5).

Now multiplying both sides of equation (8) by $(x_k - x_{0k})$ and then integrating over V, one has

$$\int_V (x_k - x_{0k})\nabla^2\left[\frac{\partial}{\partial n_0} u^*(P, P_0)\right] dV(P)$$
$$+ \int_V (x_k - x_{0k})\frac{\partial}{\partial n_0}\delta(P, P_0) dV(P) = 0. \quad (9)$$

Again, by Gauss' theorem, the first integral is

$$\int_V (x_k - x_{0k})\nabla^2\left[\frac{\partial}{\partial n_0} u^*(P, P_0)\right] dV(P)$$
$$= \int_V (x_k - x_{0k})\left(\frac{\partial u^*}{\partial n_0}\right)_{,ii} dV$$
$$= \int_V \left[(x_k - x_{0k})\left(\frac{\partial u^*}{\partial n_0}\right)_{,i}\right]_{,i} dV$$
$$- \int_V (x_k - x_{0k})_{,i}\left(\frac{\partial u^*}{\partial n_0}\right)_{,i} dV$$
$$= \int_S (x_k - x_{0k})\frac{\partial^2 u^*}{\partial n\, \partial n_0} dS - \int_S \frac{\partial u^*}{\partial n_0} n_k dS. \quad (10)$$

Then applying expression (2) to the second integral in equation (9), where $f(P) = (x_k - x_{0k})$, one obtains the third identity (6) with the result (10). This completes the operational approach where the three basic identities involving the fundamentals of the potential problem have been established. Note that these are purely mathematical properties of the fundamental solution.

A classical limit approach:

In contrast to the above, one can establish the same identities (4–6) by the usual limit approach. Consider now the 2D case (i.e. $m = 2$) for illustration. The fundamental solution for the 2D potential problem is

$$u^*(P, P_0) = \frac{1}{2\pi}\ln\frac{1}{r} \quad (11)$$

where $r = |\overrightarrow{P_0P}|$ and various directional derivatives of $u^*(P, P_0)$ are

$$\frac{\partial}{\partial n} u^*(P, P_0) = -\frac{1}{2\pi r}(\hat{r}\cdot\hat{n}) \quad (12)$$

$$\frac{\partial}{\partial n_0} u^*(P, P_0) = \frac{1}{2\pi r}(\hat{r}\cdot\hat{n}_0) \quad (13)$$

$$\frac{\partial^2}{\partial n \partial n_0} u^*(P, P_0) = \frac{1}{2\pi r^2}\left[(\hat{n}_0\cdot\hat{n}) - 2(\hat{r}\cdot\hat{n}_0)(\hat{r}\cdot\hat{n})\right] \quad (14)$$

and $\hat{r}$, $\hat{n}$ and $\hat{n}_0$ are unit vectors along the r, n and n_0 directions (Fig. 1), respectively. In the sense of ordinary functions, $u^*(P, P_0)$ and its derivatives are singular at $P = P_0$ ($r = 0$). According to the classical limit approach, a circular region $V\varepsilon(P_0)$, centered at P_0 with radius ε (which is small enough so that $V\varepsilon \subset V$ for any $P_0 \in V$), is removed. Then both sides of equation (3) are integrated over the punctured domain $V - V\varepsilon$ to obtain

$$\int_{V - V\varepsilon(P_0)} \nabla^2 u^*(P, P_0)\, dV(P) = 0, \quad \forall P_0 \in V \quad (15)$$

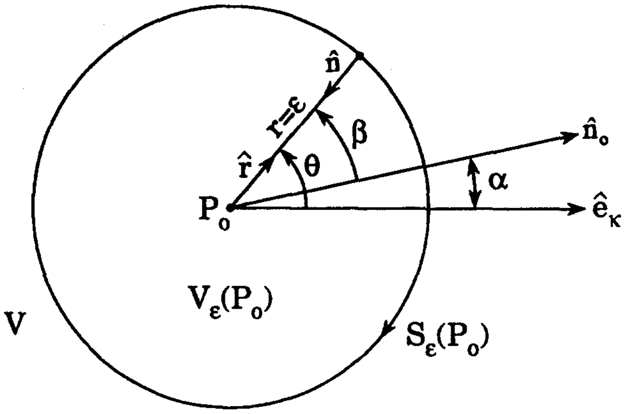

Fig. 2. Definitions of $V\varepsilon(P_0)$ and $S\varepsilon(P_0)$

where expression (1) (with V being replaced by $V - V\varepsilon$) has been applied. Since $u^*(P, P_0)$ is well behaved for any $P \in V - V\varepsilon$, one can apply Green's theorem in the classical fashion. Thus equation (15) becomes

$$\int_S \frac{\partial}{\partial n} u^*(P, P_0)\, dS(P) + \int_{S\varepsilon(P_0)} \frac{\partial}{\partial n} u^*(P, P_0)\, dS(P) = 0 \quad (16)$$

where $S\varepsilon(P_0)$ is the new boundary of V when $V\varepsilon(P_0)$ is removed, Fig. 2. Then using (12) and noticing that on $S\varepsilon$ $\hat{n} = -\hat{r}$ and $r = \varepsilon$, one has

$$\int_{S\varepsilon(P_0)} \frac{\partial}{\partial n} u^*(P, P_0)\, dS(P) = \int_{S\varepsilon} -\frac{1}{2\pi\varepsilon}(-1)\, dS = 1. \quad (17)$$

Substitution of this into equation (16) provides the first identity (4) with $P_0 \in V$.

Similarly, the integration of equation (8) over the domain $V - V\varepsilon(P_0)$ yields

$$\int_S \frac{\partial^2}{\partial n \partial n_0} u^*(P, P_0) dS(P) + \int_{S\varepsilon(P_0)} \frac{\partial^2}{\partial n \partial n_0} u^*(P, P_0)\, dS(P) = 0. \quad (18)$$

Using (14) and referring to Fig. 2 (note that the positive direction of $S\varepsilon$ is clockwise), one has

$$\int_{S\varepsilon(P_0)} \frac{\partial^2}{\partial n \partial n_0} u^*(P, P_0)\, dS(P) = \int_{S\varepsilon} \frac{1}{2\pi\varepsilon^2} (\hat{r}\cdot\hat{n}_0)\, dS = \frac{1}{2\pi\varepsilon^2} \int_{2\pi}^{0} \cos\beta(-\varepsilon d\beta) = 0. \quad (19)$$

Thus, equation (18) becomes the second identity (5) with $P_0 \in V$.

Multiplying equation (8) by $(x_k - x_{0k})$ and integrating over $V - V\varepsilon(P_0)$, the following is obtained after application of Green's theorem (cf. equations (9) and (10)):

$$\int_S (x_k - x_{0k}) \frac{\partial^2}{\partial n \partial n_0} u^*(P, P_0)\, dS(P) - \int_S n_k(P) \frac{\partial}{\partial n_0} u^*(P, P_0)\, dS(P) + \int_{S\varepsilon(P_0)} (x_k - x_{0k}) \frac{\partial^2}{\partial n \partial n_0} u^*(P, P_0)\, dS(P) - \int_{S\varepsilon(P_0)} n_k(P) \frac{\partial}{\partial n_0} u^*(P, P_0)\, dS(P) = 0. \quad (20)$$

By expression (14) and Fig. 2, the third integral in (20)

$$\int_{S\varepsilon(P_0)} (x_k - x_{0k}) \frac{\partial^2}{\partial n \partial n_0} u^*(P, P_0)\, dS(P) = \int_{S\varepsilon} (x_k - x_{0k}) \left[\frac{1}{2\pi\varepsilon^2}(\hat{r}\cdot\hat{n}_0)\right] dS = \frac{1}{2\pi\varepsilon^2}\int_{S\varepsilon} (\varepsilon\hat{r}\cdot\hat{e}_k)(\hat{r}\cdot\hat{n}_0)\, dS = \frac{1}{2\pi\varepsilon}\int_{2\pi}^{0} \cos\theta \cos(\theta - \alpha)(-\varepsilon d\theta) = \frac{1}{2}\cos\alpha = \frac{1}{2} n_{0k}(P_0). \quad (21)$$

Similarly, by expression (13) the last integral in (20)

$$\int_{S\varepsilon(P_0)} n_k(P) \frac{\partial}{\partial n_0} u^*(P, P_0)\, dS(P) = \int_{2\pi}^{0} (-\cos\theta)\left[\frac{1}{2\pi\varepsilon}\cos(\theta - \alpha)\right](-\varepsilon d\theta) = -\frac{1}{2} n_{0k}(P_0). \quad (22)$$

Substitution of (21) and (22) into equation (20) yields the third identity (6) with $P_0 \in V$. The derivations of the identities (4–6) with $P_0 \in E$ are trivial. This completes the derivations by the limit approach for the potential problem.

(B) Elastostatic problems

If u^*_{ki} denotes the fundamental solution (Kelvin) for elastostatic problems and if p^*_{ki} and σ^*_{kij} are tractions and stresses resulting from u^*_{ki}, respectively, then the stress tensor σ^*_{kij} must satisfy the equilibrium equation

$$\sigma^*_{kij,j}(P, P_0) + \delta_{ki}\delta(P, P_0) = 0, \quad \forall P, P_0 \in \mathbb{R}^m \quad (23)$$

where δ_{ki} is the Kronecker delta and $\delta_{ki}\delta(P, P_0)$ represents the volume density of the i-th component of a unit concentrated force acting at the source point $P_0 (\in \mathbb{R}^m)$ in the x_k direction. The index k in σ^*_{kij}, p^*_{ki} and u^*_{ki} indicates the direction of the unit force.

The following three identities can be readily established for this fundamental solution:

The first identity

$$\int_S p^*_{ki}(P, P_0)\, dS(P) = \begin{cases} -\delta_{ki}, & \forall P_0 \in V, \\ 0, & \forall P_0 \in E; \end{cases} \quad (24)$$

The second identity

$$\int_S \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)\, dS(P) = 0, \quad \forall P_0 \in V \cup E; \quad (25)$$

The third identity

$$\int_S E_{ilst} n_s(P) \frac{\partial}{\partial x_{0j}} u^*_{kt}(P, P_0)\, dS(P) - \int_S (x_l - x_{0l}) \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)\, dS(P) = \begin{cases} \delta_{ik}\delta_{jl}, & \forall P_0 \in V, \\ 0, & \forall P_0 \in E; \end{cases} \quad (26)$$

where E_{ijkl} is the elastic modulus tensor which relates stress σ^*_{kij} with displacement u^*_{ki} by

$$\sigma^*_{kij} = E_{ijst} u^*_{ks,t} \quad (27)$$

with $E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij}$.

The above identities (24–26) are readily derived by the operational approach which we have applied to the potential problem. The first is immediately obtained by integrating equation (23) over the closed domain V and the second identity (25) is derived by integrating the derivative equation

$$\left[\frac{\partial}{\partial x_{0j}} \sigma^*_{kis}(P, P_0)\right]_{,s} + \delta_{ki} \frac{\partial}{\partial x_{0j}} \delta(P, P_0) = 0, \quad \forall P, P_0 \in \mathbb{R}^m \quad (28)$$

over the domain V. The derivation of the third identity (26) is now carried out in some detail. Multiplying equation (28) by $(x_l - x_{0l})$ and then integrating the resulting equation over V, one has

$$\int_V (x_l - x_{0l}) \left[\frac{\partial}{\partial x_{0j}} \sigma^*_{kis}(P, P_0)\right]_{,s} dV(P) + \delta_{ki} \int_V (x_l - x_{0l}) \frac{\partial}{\partial x_{0j}} \delta(P, P_0)\, dV(P) = 0. \quad (29)$$

By Gauss' theorem (in the generalized sense) and expression (27), the first integral in equation (29) is converted to

$$\begin{aligned} \int_V (x_l - x_{0l}) \left[\frac{\partial}{\partial x_{0j}} \sigma^*_{kis}(P, P_0)\right]_{,s} dV(P) &= \int_V \left[(x_l - x_{0l}) \frac{\partial}{\partial x_{0j}} \sigma^*_{kis}\right]_{,s} dV - \int_V (x_l - x_{0l})_{,s} \frac{\partial}{\partial x_{0j}} \sigma^*_{kis}\, dV \\ &= \int_S (x_l - x_{0l}) \frac{\partial}{\partial x_{0j}} p^*_{ki}\, dS - \int_V \frac{\partial}{\partial x_{0j}} \sigma^*_{kil}\, dV \\ &= \int_S (x_l - x_{0l}) \frac{\partial}{\partial x_{0j}} p^*_{ki}\, dS - \int_V E_{ilst} \frac{\partial}{\partial x_{0j}} u^*_{kt,s}\, dV \\ &= \int_S (x_l - x_{0l}) \frac{\partial}{\partial x_{0j}} p^*_{ki}\, dS - \int_S E_{ilst} n_s \frac{\partial}{\partial x_{0j}} u^*_{kt}\, dS. \end{aligned} \quad (30)$$

By expression (2), the second integral in equation (29) is

$$\int_V (x_l - x_{0l}) \frac{\partial}{\partial x_{0j}} \delta(P, P_0)\, dV(P) = -\frac{\partial}{\partial x_{0j}} (x_l - x_{0l}) = \delta_{jl}. \quad (31)$$

Substituting (30) and (31) into equation (29), one obtains the third identity (26) with $P_0 \in V$ and this concludes the derivation of the three identities for the elastostatic fundamental solution.

It is emphasized that the manner of derivation of the identities (24–26) is not crucial. They can also be derived in a more classical way (the limit approach), as we have done for the potential problem, and other ways that will be pointed out in the following sections. What is important is the fact that these identities are properties of the fundamental solutions and can play a significant role in weakly-singular BEM formulations.

(C) Physical interpretations

Physical interpretations of the identities (24–26) for elastostatic problems are as follows, and the identities (4–6) for potential problems have similar interpretations. Identity (24) corresponds to the unit concentrated force (acting at P_0) which is balanced by the tractions on surface S, where the domain V is cut from the infinite space $\mathbb{R}^m$, if $P_0 \in V$. The resultant of the tractions on S is zero if the source point P_0 is outside the domain V ($P_0 \in E$).

The term containing the derivative of the delta function with respect to x_{0j} in equation (28) represents a unit concentrated *moment* acting at P_0. Thus the derivatives of σ^*_{kij}, p^*_{ki} and u^*_{ki} with respect to x_{0j} are the stress, traction and displacement tensors, respectively, corresponding to this unit concentrated moment. The physical meaning of the second identity (25) is that the resultant force of the traction due to this unit concentrated moment on S is a zero force regardless of where the moment is located. The third identity (26) suggests that the resultant moment of the traction on S with respect to the source point P_0 is balanced with the unit concentrated moment if $P_0 \in V$ or is zero if $P_0 \in E$, although the physical meaning of the every term in identity (26) is not readily identified.

WEAKLY-SINGULAR BIE'S AND DERIVATIVE BIE'S FOR POTENTIAL PROBLEMS

To derive weakly-singular forms of a BIE, Laplace's equation for the potential problem, i.e.,

$$\nabla^2 u(P) = 0, \quad \forall P \in \Omega \quad (32)$$

is first considered, where $\Omega \subset \mathbb{R}^m$ is the physical domain in which a solution is desired. The boundary of Ω is denoted by Γ (Ω can be closed or infinite) as shown in Fig. 3.

By means of the fundamental solution defined by equation (3) and Green's second identity (the reciprocal theorem for the Laplacian operator), the solution of equation (32) can be expressed by the representation

$$u(P_0) = \int_\Gamma \left[u^*(P, P_0) \frac{\partial}{\partial n} u(P) - u(P) \frac{\partial}{\partial n} u^*(P, P_0)\right] d\Gamma(P), \quad \forall P_0 \in \Omega. \quad (33)$$

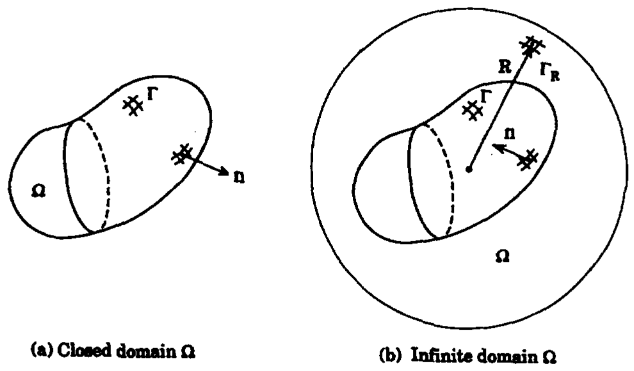

Fig. 3. Two kinds of domains

The directional derivative of the representation at P_0 in direction $n_0(P_0)$ is

$$\frac{\partial}{\partial n_0} u(P_0) = \int_\Gamma \left[\frac{\partial}{\partial n_0} u^*(P, P_0) \frac{\partial}{\partial n} u(P) - u(P) \frac{\partial^2}{\partial n \partial n_0} u^*(P, P_0) \right] d\Gamma(P), \quad \forall P_0 \in \Omega. \quad (34)$$

It is interesting to note that the identities (4–6) developed in the previous section can be derived from the above two representations *if* Ω is a closed domain. First, if one chooses $u(P) = 1, \forall P \in \Omega \cup \Gamma$ (a particular solution of equation (32)) in (33) and (34), one then obtains the first identity (4) and the second identity (5), respectively. Further, if $u(P) = (x_k - x_{0k}), \forall P \in \Omega \cup \Gamma$ (another particular solution of equation (32)) in (34), the third identity (6) results.

To derive the weakly-singular BIE from the representation (33), let P_0 tend to the boundary Γ. This can be done by putting P_0 on Γ first and then shrinking the small surface $\Gamma\varepsilon(P_0)$ which is the part outside the domain Ω of a sphere (or a circle if $m = 2$) with radius ε and center P_0, Fig. 4. Before the limit is evaluated, one has[2,3] from (33)

$$u(P_0) = \lim_{\varepsilon \to 0} \int_{\Gamma_c} \left[u^*(P, P_0) \frac{\partial}{\partial n} u(P) - u(P) \frac{\partial}{\partial n} u^*(P, P_0) \right] d\Gamma(P) + \lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon(P_0)} \left[u^*(P, P_0) \frac{\partial}{\partial n} u(P) - u(P) \frac{\partial}{\partial n} u^*(P, P_0) \right] d\Gamma(P), \quad \forall P_0 \in \Gamma \quad (35)$$

where Γ_c is the remaining part of Γ outside the sphere centered at P_0. It is easily verified that the limit of the first integral on Γε vanishes, i.e.,

$$\lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon(P_0)} u^*(P, P_0) \frac{\partial}{\partial n} u(P)\, d\Gamma(P) = 0. \quad (36)$$

Instead of evaluating the limit of the second integral on Γε of equation (35), one can convert it into the limit of an integral on Γ_c.

If Ω is a *closed* domain, then by applying the first identity (4) with V being the domain enclosed by the surface $S = \Gamma_c \cup \Gamma\varepsilon(P_0)$, one has

$$\int_{\Gamma\varepsilon(P_0)} \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P) = -1 - \int_{\Gamma c} \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P). \quad (37)$$

Thus by the mean value theorem and then the use of (37), the second integral on Γε of equation (35) is converted to

$$\lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon(P_0)} u(P) \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P) = \lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon(P_0)} \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P) u(P_0) = -u(P_0) - \lim_{\varepsilon \to 0} \int_{\Gamma c} \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P) u(P_0). \quad (38)$$

By virtue of expressions (36) and (38), one obtains the following weakly-singular form of the BIE for the closed domain from (35)

$$\int_\Gamma \frac{\partial}{\partial n} u^*(P, P_0) \left[u(P) - u(P_0) \right] d\Gamma(P) = \int_\Gamma u^*(P, P_0) \frac{\partial}{\partial n} u(P)\, d\Gamma(P), \quad \forall P_0 \in \Gamma \quad (39)$$

where $\int_\Gamma [\ ] d\Gamma \equiv \lim_{\varepsilon \to 0} \int_{\Gamma c} [\ ] d\Gamma$ and the integral on the left hand side does not exhibit the $O(1/r)$ (for $m = 2$) or $O(1/r^2)$ (for $m = 3$) singularity *if* $u(P)$ is continuous at $P_0 \in \Gamma$.

If Ω is an *infinite* domain, one can apply the second part of the first identity (4), i.e. with E being the domain outside the surface $S = \Gamma_c \cup \Gamma\varepsilon(P_0)$, to obtain

$$\int_{\Gamma\varepsilon(P_0)} \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P) = -\int_{\Gamma c} \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P). \quad (40)$$

This can also be obtained by applying the first part of the identity (4) with V being the domain enclosed by $S = \Gamma_R \cup \Gamma_c \cup \Gamma\varepsilon(P_0)$, where Γ_R is the surface of a sufficiently large sphere (or a circle if $m = 2$) with radius R, see Fig. 3(b). Thus by the use of (40), the second integral on Γε of equation (35) becomes

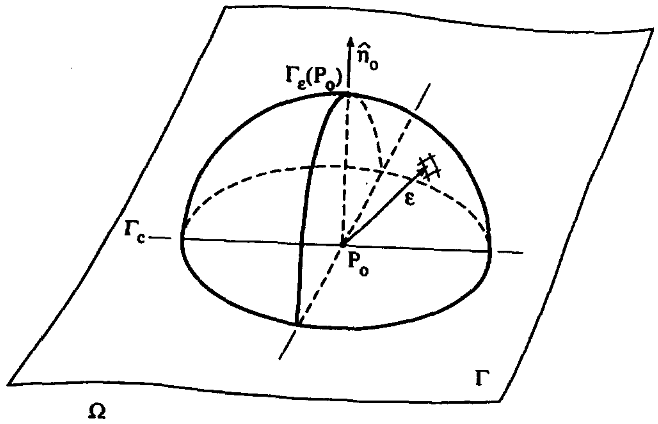

Fig. 4. Definitions of $\Gamma\varepsilon(P_0)$ and Γc

$$\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon(P_0)} u(P)\frac{\partial}{\partial n}u^*(P,P_0)\,d\Gamma(P)$$
$$= -\lim_{\varepsilon\to 0}\int_{\Gamma c}\frac{\partial}{\partial n}u^*(P,P_0)\,d\Gamma(P)u(P_0). \quad (41)$$

Substitutions of (36) and (41) into (35) yield the following weakly-singular BIE for the infinite domain

$$u(P_0)+\int_\Gamma \frac{\partial}{\partial n}u^*(P,P_0)\left[u(P)-u(P_0)\right]d\Gamma(P)$$
$$=\int_\Gamma u^*(P,P_0)\frac{\partial}{\partial n}u(P)\,d\Gamma(P),\quad \forall P_0\in\Gamma. \quad (42)$$

To derive the weakly-singular derivative BIE from the representation (34), let P_0 tend to Γ following the same procedure as for the above ordinary BIE and with n_0 taken as the outward normal at $P_0\in\Gamma$ (Fig. 4). Before the limit is evaluated, one has, from (34),

$$\frac{\partial}{\partial n_0}u(P_0)=\lim_{\varepsilon\to 0}\int_{\Gamma c}\left[\frac{\partial}{\partial n_0}u^*(P,P_0)\frac{\partial}{\partial n}u(P)\right.$$
$$\left.-u(P)\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)\right]d\Gamma(P)$$
$$+\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon}\left[\frac{\partial}{\partial n_0}u^*(P,P_0)\frac{\partial}{\partial n}u(P)\right.$$
$$\left.-u(P)\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)\right]d\Gamma(P),\quad \forall P_0\in\Gamma. \quad (43)$$

If Ω is a *closed* domain, then by the mean value theorem and the use of the third identity (6), with V being the domain enclosed by $S=\Gamma_c\cup\Gamma\varepsilon$, the first integral on $\Gamma\varepsilon$ is

$$\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon}\frac{\partial}{\partial n_0}u^*(P,P_0)\frac{\partial}{\partial n}u(P)d\Gamma(P)$$
$$=\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon}\frac{\partial}{\partial n_0}u^*(P,P_0)n_k(P)\,d\Gamma(P)u_{,k}(P_0)$$
$$=\lim_{\varepsilon\to 0}\left[n_{0k}(P_0)+\int_{\Gamma c\cup\Gamma\varepsilon}(x_k-x_{0k})\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)d\Gamma(P)\right.$$
$$\left.-\int_{\Gamma c}n_k(P)\frac{\partial}{\partial n_0}u^*(P,P_0)\,d\Gamma(P)\right]u_{,k}(P_0)$$
$$=\frac{\partial}{\partial n_0}u(P_0)+\lim_{\varepsilon\to 0}\int_{\Gamma_c\cup\Gamma\varepsilon}\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)$$
$$\times[u_{,k}(P_0)(x_k-x_{0k})]\,d\Gamma(P)$$
$$-\lim_{\varepsilon\to 0}\int_{\Gamma c}\frac{\partial}{\partial n_0}u^*(P,P_0)[u_{,k}(P_0)n_k(P)]\,d\Gamma(P). \quad (44)$$

For the second integral on $\Gamma\varepsilon$ of equation (43), the density function $u(P)$ is expanded about P_0 using Taylor's theorem, and the second identity (5) is invoked to provide

$$\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon}u(P)\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)\,d\Gamma(P)$$
$$=\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon}\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)[u(P_0)+u_{,k}(P_0)(x_k-x_{0k})$$
$$+O(\varepsilon^2)]\,d\Gamma(P)$$
$$=-\lim_{\varepsilon\to 0}\int_{\Gamma c}\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)[u(P_0)]\,d\Gamma(P)$$
$$+\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon}\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)$$
$$\times[u_{,k}(P_0)(x_k-x_{0k})]\,d\Gamma(P) \quad (45)$$

where it is noted that

$$\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon}\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)\,O(\varepsilon^2)\,d\Gamma(P)=0.$$

This latter observation is easily verified by considering the orders of $(\partial^2/\partial n\partial n_0)u^*(P,P_0)$ and the area (or length) of $\Gamma\varepsilon$ for 3D (or 2D) problems.

Substituting (44) and (45) into equation (43), one finds that the integrals on $\Gamma\varepsilon$ cancel to leave the following weakly-singular derivative BIE for the closed domain

$$\int_\Gamma\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)[u(P)-u(P_0)$$
$$-u_{,k}(P_0)(x_k-x_{0k})]\,d\Gamma(P)$$
$$=\int_\Gamma\frac{\partial}{\partial n_0}u^*(P,P_0)[u_{,k}(P)-u_{,k}(P)]n_k(P)\,d\Gamma(P),$$
$$\forall P_0\in\Gamma \quad (46)$$

in which the integral $\int_\Gamma[\ ]d\Gamma\equiv\lim_{\varepsilon\to 0}\int_{\Gamma c}[\ ]d\Gamma$.

Again, as for the ordinary BIE of equation (39), the limit interpretation of this integral equation as an improper one is not required, since both integrals are regular, with at most weakly singular integrands, as $P\to P_0$. This weakly-singular derivative BIE for potential problems in a closed domain was first presented for the 2D case in Ref. 6 without using the identities developed in this paper and was derived by Rudolphi[10] by imposing simple solutions onto the limit form of the representation. From the derivation and the final form of equation (46), it is noticed that the *derivative* of the function $u(P)$ is required to be continuous at $P_0\in\Gamma$, which is a stronger condition on equation (46) than that on equation (39).

If Ω is an *infinite* domain, the applications of the second identity (5) and the third identity (6) will provide the following weakly-singular derivative BIE from equation (43) for the infinite domain,

$$\frac{\partial}{\partial n_0}u(P_0)+\int_\Gamma\frac{\partial^2}{\partial n\partial n_0}u^*(P,P_0)[u(P)-u(P_0)$$
$$-u_{,k}(P_0)(x_k-x_{0k})]\,d\Gamma(P)$$
$$=\int_\Gamma\frac{\partial}{\partial n_0}u^*(P,P_0)\left[u_{,k}(P)-u_{,k}(P_0)\right]n_k(P)\,d\Gamma(P),$$
$$\forall P_0\in\Gamma. \quad (47)$$

To the authors' knowledge, the weakly-singular BIE (42) and the weakly-singular derivative BIE (47) for the infinite domain have not been reported in the literature.

In the BEM literature, the ordinary BIE is conventionally written in the following *singular* form

$$C(P_0)u(P_0) = \int_\Gamma \left[u^*(P, P_0) \frac{\partial}{\partial n} u(P) - u(P) \frac{\partial}{\partial n} u^*(P, P_0) \right] d\Gamma(P), \quad \forall P_0 \in \Gamma \quad (48)$$

where the integral of the second term on the right hand side is a CPV and the coefficient $C(P_0)$ is given by (cf. equations (35) and (36))

$$C(P_0) = 1 + \lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon(P_0)} \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P), \; \forall P_0 \in \Gamma. \quad (49)$$

If Γ is smooth at P_0, then the evaluation of expression (49) provides $C(P_0) = 1/2$. The singular BIE (48) with $C(P_0)$ defined by (49) is valid for both closed and infinite domains. With the applications of the first identity, or equations (37) and (40), the expression (49) for $C(P_0)$ can be converted to

$$C(P_0) = -\int_\Gamma \frac{\partial}{\partial n} u^*(P, P_0)\, d\Gamma(P), \quad \forall P_0 \in \Gamma \quad (50)$$

for a closed domain and

$$C(P_0) = 1 - \int_\Gamma \frac{\partial}{\partial n_0} u^*(P, P_0)\, d\Gamma(P), \quad \forall P_0 \in \Gamma \quad (51)$$

for an infinite domain, respectively, where the integral is a CPV. Substitutions of (50) and (51) into the singluar BIE (48) will, again, yield the weakly-singular forms (39) and (42) of the BIE for the closed and infinite domains, respectively. This equivalence, or transformation from a singular form to a weakly-singular one, clearly shows that the BIE, in nature, is regular as demonstrated more than a decade ago in Ref. 4 in the context of elastostatics.

WEAKLY-SINGULAR BIE'S AND DERIVATIVE BIE'S FOR ELASTOSTATIC PROBLEMS

The development of weakly-singular forms for the BIE's and the derivative BIE'S for elastostatic problems parallels the previous development for potential problems. In the absence of body forces, the equilibrium equations, in terms of stress tensor σ_{ij}, are

$$\sigma_{ij,j}(P) = 0, \quad \forall P \in \Omega \quad (52)$$

where $\Omega \subset \mathbb{R}^m$ as in Fig. 3. By means of the fundamental solution defined by equation (23) and Betti's identity in elasticity, the solution of equation (52) can be expressed by the following integral representation[2,3,4]

$$u_k(P_0) = \int_\Gamma u^*_{ki}(P, P_0) p_i(P)\, d\Gamma(P) - \int_\Gamma p^*_{ki}(P, P_0) u_i(P)\, d\Gamma(P), \quad \forall P_0 \in \Omega \quad (53)$$

where u_i and p_i are displacement and traction components respectively. The gradient of the displacement field is given by

$$\frac{\partial}{\partial x_{0j}} u_k(P_0) = \int_\Gamma \frac{\partial}{\partial x_{0j}} u^*_{ki}(P, P_0) p_i(P)\, d\Gamma(P) - \int_\Gamma \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0) u_i(P)\, d\Gamma(P), \quad \forall P_0 \in \Omega. \quad (54)$$

If Ω is a closed domain, then expressions (53) and (54) can be used to derive the three identities (24–26). First with the choice $u_i(P) = 1$ ($\forall P \in \Omega \cup \Gamma$) and substitutions into (53) and (54), one obtains (24) and (25) with $P_0 \in V$ ($=\Omega$), respectively. Then with $u_i(P) = x_i - x_{0i}$ ($\forall P \in \Omega \cup \Gamma$) in (54), one obtains (26) with $P_0 \in V$.

Now by letting P_0 tend to Γ in representation (53) as was done for the potential problem (Fig. 4), one has

$$\begin{aligned} u_k(P_0) = {} & \lim_{\varepsilon \to 0} \int_{\Gamma c} u^*_{ki}(P, P_0) p_i(P)\, d\Gamma(P) - \lim_{\varepsilon \to 0} \int_{\Gamma c} p^*_{ki}(P, P_0) u_i(P)\, d\Gamma(P) \\ & + \lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon} u^*_{ki}(P, P_0) p_i(P)\, d\Gamma(P) - \lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon} p^*_{ki}(P, P_0) u_i(P)\, d\Gamma(P), \quad \forall P_0 \in \Gamma. \quad (55) \end{aligned}$$

Similarly to the derivation for the potential problem, the limit of the first integral on $\Gamma\varepsilon$ vanishes. With the applications of the first identity (24) and the mean value theorem, the limit of the second integral on $\Gamma\varepsilon$ is converted to

$$\lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon} p^*_{ki}(P, P_0) u_i(P)\, d\Gamma(P) = -u_k(P_0) - \lim_{\varepsilon \to 0} \int_{\Gamma c} p^*_{ki}(P, P_0)\, d\Gamma(P)\, u_i(P_0) \quad (56)$$

for a closed domain and

$$\lim_{\varepsilon \to 0} \int_{\Gamma\varepsilon} p^*_{ki}(P, P_0) u_i(P)\, d\Gamma(P) = -\lim_{\varepsilon \to 0} \int_{\Gamma c} p^*_{ki}(P, P_0)\, d\Gamma(P) u_i(P_0) \quad (57)$$

for an infinite domain, respectively. Substitutions of (56) and (57) into equation (55) yield the following weakly-singular BIE's, i.e.

$$\int_\Gamma p^*_{ki}(P, P_0)[u_i(P) - u_i(P_0)]\, d\Gamma(P) = \int_\Gamma u^*_{ki}(P, P_0) p_i(P)\, d\Gamma(P), \quad \forall P_0 \in \Gamma \quad (58)$$

for the closed domain, and

$$u_k(P_0) + \int_\Gamma p^*_{ki}(P, P_0)[u_i(P) - u_i(P_0)]\, d\Gamma(P) = \int_\Gamma u^*_{ki}(P, P_0) p_i(P)\, d\Gamma(P), \; \forall P_0 \in \Gamma \quad (59)$$

for the infinite domain, respectively. The integral on the left hand side of equation (58) or equation (59) is not a strongly singular (Cauchy-type) integral since the displacement u_i is continuous by the assumption in elasticity.

To obtain the weakly-singular derivative BIE's based on expression (54), let P_0 tend to Γ in equation (54) (see Fig. 4), i.e.

$$\frac{\partial}{\partial x_{0j}} u_k(P_0) = \lim_{\varepsilon\to 0} \int_{\Gamma c} \frac{\partial}{\partial x_{0j}} u^*_{ki}(P, P_0)p_i(P)\, d\Gamma(P) - \lim_{\varepsilon\to 0} \int_{\Gamma c} \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)u_i(P)\, d\Gamma(P) + \lim_{\varepsilon\to 0} \int_{\Gamma\varepsilon} \frac{\partial}{\partial x_{0j}} u^*_{ki}(P, P_0)p_i(P)\, d\Gamma(P) - \lim_{\varepsilon\to 0} \int_{\Gamma\varepsilon} \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)u_i(P)\, d\Gamma(P), \quad \forall P_0 \in \Gamma. \tag{60}$$

With the applications of the second identity (25) and the third identity (26), while using Taylor's theorem to expand $u_i(P)$, the last two integrals (on $\Gamma\varepsilon$) in (60) are found to be

$$\lim_{\varepsilon\to 0} \int_{\Gamma\varepsilon} \frac{\partial}{\partial x_{0j}} u^*_{ki}(P, P_0)p_i(P)\, d\Gamma(P) = \frac{\partial}{\partial x_{0j}} u_k(P_0) + \lim_{\varepsilon\to 0} \int_{\Gamma c \cup \Gamma\varepsilon} \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)[u_{i,l}(P_0)(x_l - x_{0l})]\, d\Gamma(P) - \lim_{\varepsilon\to 0} \int_{\Gamma c} \frac{\partial}{\partial x_{0j}} u^*_{ki}(P, P_0)[\sigma_{il}(P_0)n_l(P)]\, d\Gamma(P) \tag{61}$$

and

$$\lim_{\varepsilon\to 0} \int_{\Gamma\varepsilon} \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)\, u_i(P)\, d\Gamma(P) = -\lim_{\varepsilon\to 0} \int_{\Gamma c} \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)[u_i(P_0)]\, d\Gamma(P) + \lim_{\varepsilon\to 0} \int_{\Gamma\varepsilon} \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)[u_{i,l}(P_0)\,(x_l - x_{0l})]\, d\Gamma(P) \tag{62}$$

respectively, for a *closed* domain. Substitution of these results into equation (60) yields the following weakly-singular derivative BIE for the closed domain

$$\int_\Gamma \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0)[u_i(P) - u_i(P_0) - u_{i,l}(P_0)(x_l - x_{0l})]\, d\Gamma(P) = \int_\Gamma \frac{\partial}{\partial x_{0j}} u^*_{ki}(P, P_0)[\sigma_{il}(P) - \sigma_{il}(P_0)]n_l(P)\, d\Gamma(P), \quad \forall P_0 \in \Gamma. \tag{63}$$

Similarly, the weakly-singular derivative BIE for an infinite domain is found to be

$$\frac{\partial}{\partial x_{0j}} u_k(P_0) + \int_\Gamma \frac{\partial}{\partial x_{0j}} p^*_{ki}(P, P_0) [u_i(P) - u_i(P_0) - u_{i,l}(P_0)(x_l - x_{0l})]\, d\Gamma(P) = \int_\Gamma \frac{\partial}{\partial x_{0j}} u^*_{ki}(P, P_0)[\sigma_{il}(P) - \sigma_{il}(P_0)]n_l(P)\, d\Gamma(P), \quad \forall P_0 \in \Gamma. \tag{64}$$

Equations (63) and (64) are the weakly-singular forms of the derivative BIE for elastostatic problems similar to equations (46) and (47) for potential problems. It is noted that, for the integrals to be regular, the stress field $\sigma_{ij}(P)$ must be continuous at $P_0 \in \Gamma$.

The weakly-singular equations (59), (63) and (64) have not been reported in the literature.

The ordinary BIE for the elastostatic problem is conventionally written in to following *singular* form

$$C_{ki}(P_0)u_i(P_0) = \int_\Gamma u^*_{ki}(P, P_0)p_i(P)\, d\Gamma(P) - \int_\Gamma p^*_{ki}(P, P_0)u_i(P)\, d\Gamma(P), \quad \forall P_0 \in \Gamma \tag{65}$$

where the second integral on the right hand side is a CPV and the coefficient $C_{ki}(P_0)$ is given by

$$C_{ki}(P_0) = \delta_{ki} + \lim_{\varepsilon\to 0} \int_{\Gamma\varepsilon(P_0)} p^*_{ki}(P, P_0) d\Gamma(P), \quad \forall P_0 \in \Gamma \tag{66}$$

in which $\Gamma\varepsilon(P_0)$ is defined as before (Fig. 4). If Γ is smooth at point P_0, $C_{ki}(P_0) = \frac{1}{2}\delta_{ki}$. The singular BIE (65) with $C_{ki}(P_0)$ defined by (66) is valid for both closed and infinite domains and is the most popular form in the BEM literature for elastostatic problems. By applying the first identity (24) one can write $C_{ki}(P_0)$ in (66) as

$$C_{ki}(P_0) = -\int_\Gamma p^*_{ki}(P, P_0)\, d\Gamma(P), \quad \forall P_0 \in \Gamma \tag{67}$$

for the closed domain, and

$$C_{ki}(P_0) = \delta_{ki} - \int_\Gamma p^*_{ki}(P, P_0)\, d\Gamma(P), \quad \forall P_0 \in \Gamma \tag{68}$$

for the infinite domain, respectively, where the integral is a CPV. Notice that expression (67) can also be obtained from equation (65) directly by introducing a rigid body translation, e.g. setting $u_i(P) = 1(\forall P \in \Omega \cup \Gamma)$. The result obtained in this way was first reported in Ref. 11. However, it is pointed out that, for an infinite domain, rigid body translations cannot be imposed directly onto the original BIE (65). In deriving equation (65) for an infinite domain, it is assumed that the two boundary integrals over Γ_R (as mentioned in the previous section, see Fig. 3 (b)) tend to zeros as the radius R tends to infinity. This condition would be violated if a rigid body translation were imposed. Thus modifications must be made to include the two integrals over Γ_R in (65) before introducing the rigid body translation for the infinite domain.

Substitutions of expressions (67) and (68) into equation (65) will provide the weakly-singular BIE's (58) and (59), respectively. Expressing the BIE for elastostatic problems in the form of equation (58) in this way was first introduced in Ref. 4 where the equation is arranged in a slightly different way.

The weakly-singular BIE and the weakly-singular derivative BIE will result in weakly-singular boundary element formulations as developed in the following section.

WEAKLY-SINGULAR BOUNDARY ELEMENT FORMULATIONS

The boundary element formulation based on the weakly-singular BIE's (58) and (59) for elastostatic problems is demonstrated in this section. The equivalence of this formulation to that obtained by applying the rigid displacement method in the traditional boundary element formulation is emphasized and discussed. Results presented in this section are independent of the boundary elements employed (constant, linear, quadratic, or higher order), since the weakly-singular features of the formulations are identical.

Let the boundary Γ of the domain Ω be discretized into a total of M surface elements (for $m = 3$) or line elements (for $m = 2$) and let N be the total number of the boundary nodes. Matrix notation is employed in this section in which the subscripts i, j,... no longer refer to the coordinate directions, but to global node numbers. Now suppose that the source point P_0 is placed at the i-th node on the boundary ($i = 1,2,\ldots,N$). Then following the standard procedure (see, e.g. Ref. 2–4), one can write the discretized form of equation (58) as

$$[\hat{\mathbf{H}}_{i1} \ldots \hat{\mathbf{H}}_{ii} \ldots \hat{\mathbf{H}}_{iN}] \begin{Bmatrix} \mathbf{u}_1 - \mathbf{u}_i \\ \vdots \\ \mathbf{u}_i - \mathbf{u}_i \\ \vdots \\ \mathbf{u}_N - \mathbf{u}_i \end{Bmatrix} = [\mathbf{G}_{i1} \ldots \mathbf{G}_{ii} \ldots \mathbf{G}_{iN}] \begin{Bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_i \\ \vdots \\ \mathbf{p}_N \end{Bmatrix}$$

where $\mathbf{u}_i$ and $\mathbf{p}_i$ are the displacement and traction vectors at the i-th node ($i = 1, 2, \ldots, N$), respectively (each has 3 components for $m = 3$, or 2 for $m = 2$), and the 3×3 matrices $\hat{\mathbf{H}}_{ij}$ and $\mathbf{G}_{ij}$ (or 2×2 matrices if $m = 2$) are summations of the integrals with traction kernel and displacement kernel over the elements of which the j-th node is a local node. The matrix $\hat{\mathbf{H}}_{ii}$ is a CPV since $j = i$ means that the source point i is on the elements where the integrals are calculated. This matrix, $\hat{\mathbf{H}}_{ii}$, is eliminated from the boundary element formulation.

The vectors $\mathbf{u}_j - \mathbf{u}_i$ ($j = 1,2,\ldots,N$) in the above equation can be regarded as new displacement vectors (the relative displacement of the j-th node to the i-th node for fixed i). Rearranging this equation, one has

$$[\hat{\mathbf{H}}_{i1} \ldots \hat{\mathbf{H}}_{ii} \ldots \hat{\mathbf{H}}_{iN}] \begin{Bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_N \end{Bmatrix} + \left(- \sum_{j=1}^{N} \hat{\mathbf{H}}_{ij} \right) \mathbf{u}_i = [\mathbf{G}_{i1} \ldots \mathbf{G}_{ii} \ldots \mathbf{G}_{iN}] \begin{Bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_i \\ \vdots \\ \mathbf{p}_N \end{Bmatrix}$$

Combining the two terms on the left hand side, one obtains

$$\left[\hat{\mathbf{H}}_{i1} \ldots \left(- \sum_{j \neq i}^{N} \hat{\mathbf{H}}_{ij} \right) \ldots \hat{\mathbf{H}}_{iN} \right] \begin{Bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_N \end{Bmatrix} = [\mathbf{G}_{i1} \ldots \mathbf{G}_{ii} \ldots \mathbf{G}_{iN}] \begin{Bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_i \\ \vdots \\ \mathbf{p}_N \end{Bmatrix} \quad (69)$$

where $i = 1, 2, \ldots, N$. Notice that $\hat{\mathbf{H}}_{ii}$ has been eliminated. If one writes all these equations together, one obtains the following system of equations

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \ldots & \mathbf{H}_{1N} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \ldots & \mathbf{H}_{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{H}_{N1} & \mathbf{H}_{N2} & \ldots & \mathbf{H}_{NN} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} & \ldots & \mathbf{G}_{1N} \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \ldots & \mathbf{G}_{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{G}_{N1} & \mathbf{G}_{N2} & \ldots & \mathbf{G}_{NN} \end{bmatrix} \begin{Bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_N \end{Bmatrix} \quad (70)$$

or

$$\mathbf{Hu} = \mathbf{Gp}$$

where $\mathbf{H}$ and $\mathbf{G}$ are $3N \times 3N$ (or $2N \times 2N$ if $m = 2$) matrices, $\mathbf{u}$ and $\mathbf{p}$ are vectors of $3N$ (or $2N$) components, and the submatrices of $\mathbf{H}$ are given by

$$\mathbf{H}_{ii} = - \sum_{j \neq i}^{N} \mathbf{H}_{ij} \quad (71)$$

$$\mathbf{H}_{ij} = \hat{\mathbf{H}}_{ij}, \quad \text{for} \quad j \neq i \quad (72)$$

with $i, j = 1, 2, \ldots, N$. Expression (71) shows that the diagonal submatrices of the $\mathbf{H}$ matrix in equation (70) are determined by the off-diagonal submatrices in the same row and therefore the calculation of the CPV in $\hat{\mathbf{H}}_{ii}$ is eliminated. The above result is valid for the closed domain.

Comparing the weakly-singular BIE's (58) and (59) for closed and infinite domains, respectively, we notice that the only difference is the additional free displacement term in (59). Thus, for an infinite domain the discretization scheme employed above, with a slight modification, will yield the following counterpart of equation (69)

$$\left[\hat{\mathbf{H}}_{i1} \ldots \left(\mathbf{I} - \sum_{j \neq i}^{N} \hat{\mathbf{H}}_{ij} \right) \ldots \hat{\mathbf{H}}_{iN} \right] \begin{Bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_N \end{Bmatrix} = [\mathbf{G}_{i1} \ldots \mathbf{G}_{ii} \ldots \mathbf{G}_{iN}] \begin{Bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_i \\ \vdots \\ \mathbf{p}_N \end{Bmatrix} \quad (73)$$

where $i = 1, 2, \ldots, N$ and $\mathbf{I}$ is the 3×3 (or 2×2 if $m = 2$) idenitity matrix. And the final system of equations for an infinite domain will be of the same form as that of equation (70) in which only the diagonal submatrices of the $\mathbf{H}$ matrix are modified as follows

$$\mathbf{H}_{ii} = \mathbf{I} - \sum_{j \neq i}^{N} \mathbf{H}_{ij} \quad (74)$$

for $i = 1, 2, \ldots, N$.

Thus the weakly-singular boundary element forulation, represented by equation (70) with submatrices $\mathbf{H}_{ii}$ given by (71) for a closed domain or (74) for an infinite domain, is established based on the weakly-singular boundary integral equation (58) or (59), respectively. The procedure to obtain this result is mathematically strict and no new assumptions or approximations are introduced. The advantages of this formulation are obvious. First, there is no computation if the CPV's in this formulation which will eliminate the error that exists if the CPV's are obtained by numerical integrations. Secondly, the coefficient $C_{kl}(P_0)$ need not to be calculated in any sense (analytically or numerically), which can save a great deal of effort for a non-smooth surface in 3D problems. All the benefits from this formulation will make BEM computer programs simpler and more efficient.

The weakly-singular boundary element formulation in equation (70) with expressions (71) and (74) is identical to that of the rigid body displacement method[2,3] (the third choice to deal with the CPV's mentioned in the introduction section). Thus, it is believed that this paper has provided a mathematical verification for this method which is based on the discretized BEM formulation. It is also suggested that the implementation of the physical concept (the rigid body translation) is in agreement with the mathematical interpretation (the weakly-singular BIE) in the BEM formulation. This agreement is embedded in the boundary integral equation and the boundary element formulation. In fact, the first author has shown directly[12], without applying the weakly-singular BIE, that the diagonal submatrices $\mathbf{H}_{ii}$ obtained by using the rigid body displacement method is exactly equal to those obtained by calculating the CPV's and the **C** matrices analytically on the same kind of elements. All these show that the calculations of the CPV's (for boundary integrals) and the **C** matrices are unnecessary in the BEM.

A similar weakly-singular boundary element formulation of the weakly-singular derivative BIE's in which no CPV's or HFP's are involved, can be developed through a discretization procedure similar to that demonstrated in this section, although more manipulations are required.

CONCLUSIONS

Three integral identities for each of the fundamental solutions of potential and elastostatic problems are established in this paper. These identities can be employed in the derivation of the weakly-singular BIE's and weakly-singular derivative BIE's in a systematic way. The boundary element formulation, based on the weakly-singular BIE's, is shown to lead to the same expressions for the diagonal submatrices $\mathbf{H}_{ii}$ as that of the rigid body displacement method when applied to the discretized equations of the conventional singular BIE. Thus the rigid body displacement method is shown to be an exact approach to determine the diagonal submatrices. The practice of calculating the CPV's in conventional BIE's or the HFP's in conventional derivative BIE's and the geometric matrices **C**, either analytically or numerically, *cannot* be expected to give better numerical solutions than the weakly-singular boundary element formulation shown herein.

REFERENCES

1 Cerrolaza, M. and Alarcon, E. A bi-cubic transformation for the numerical evaluation of the Cauchy principal value integrals in boundary methods, *Int. J. Numer. Methods Eng.*, 1989, **28**, 987–999

2 Brebbia, C. A. *The Boundary Element Method for Engineers*, Pentech Press, London, 1978

3 Brebbia, C. A. and Dominguez, J. *Boundary Elements an Introductory Course* (Chapter 3), McGraw-Hill, New York, 1989

4 Rizzo, F. J. and Shippy, D. J. An advanced boundary integral equation method for three-dimensional thermoelasticity, *Int. J. Numer, Methods Eng.*, 1977, **11**, 1753–1768

5 Rizzo, F. J., Shippy, D. J. and Rezayat, M. A boundary integral equation method for radiation and scattering of elastic waves in three dimensions, *Int. J. Numer. Methods Eng.*, 1985, **21**, 115–129

6 Rudolphi, T. J., Krishnasamy, G., Schmerr, L. W. and Rizzo, F. J. On the use of strongly singular integral equations for crack problems, *Boundary Elements X*, ed, C. A. Brebbia, Springer-Verlag and Comp. Mech. Publications, Southampton, England, 1988, 249–263

7 Krishnasamy, G., Rudolphi, T. J., Schmerr, L. W. and Rizzo, F. J. Hypersingular boundary integral equations: some applications in acoustic and elastic wave scattering, *J. of Applied Mechanics*, 1990, **57**, 404–414

8 Zemanian, A. H. *Distribution Theory and Transform Analysis – An Introduction to Generalized Functions, with Applications*, Dover, New York. 1987

9 Gipson, G. S. *Boundary Element Fundamentals – Basic Concepts and Recent Developments in The Poisson Equation*, Computational Mechanics Publications, Southampton, 1987

10 Rudolphi, T. J. The use of simple solutions in the regularization of hypersingular boundary integral equations, *Mathl Comput. Modelling*, 1991, **15**, No. 3–5, 269–278

11 Cruse, T. A. An improved boundary-integral equation method for three dimensional elastic stress analysis, *Computers & Structures*, 1974, **4**, 741–754

12 Liu, Y. J. On the equivalence of the rigid body displacement method and the method of calculating CPV's and **C** matrix analytically in determining the diagonal submatrices $\mathbf{H}_{ii}$ (in Chinese), manuscript, 1986, unpublished