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On the conventional boundary integral equation formulation for piezoelectric solids with defects or of thin shapes

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Abstract

In this paper, the conventional boundary integral equation (BIE) formulation for piezoelectric solids is revisited and the related issues are examined. The key relations employed in deriving the piezoelectric BIE, such as the generalized Green's identity (reciprocal work theorem) and integral identities for the piezoelectric fundamental solution, are established rigorously. A weakly singular form of the piezoelectric BIE is derived for the first time using the identities for the fundamental solution, which eliminates the calculation of any singular integrals in the piezoelectric boundary element method (BEM). The crucial question of whether or not the piezoelectric BIE will degenerate when applied to crack and thin shell-like problems is addressed. It is shown analytically that the conventional BIE for piezoelectricity does degenerate for crack problems, but does *not* degenerate for thin piezoelectric shells. The latter has significant implications in applications of the piezoelectric BIE to the analysis of thin piezoelectric films used widely as sensors and actuators. Numerical tests to show the degeneracy of the piezoelectric BIE for crack problems are presented and one remedy to this degeneracy by using the multi-domain BEM is also demonstrated.

Keywords: Piezoelectric solids; Boundary integral equation; Identities; Cracks; Thin shells

1. Introduction

Piezoelectric materials have been used widely as sensors and actuators in smart materials or structures because they have many desirable properties [1–3]. Simulations of piezoelectric solids, on the other hand, are very challenging because of the anisotropy in piezoelectric materials, coupling of elastic and electric fields and thinness of the piezoelectric devices (for example, the thickness of sensors/actuators is in the range of a few μm to a few hundred μm). To add to the level of difficulty, the simulation of piezoelectric sensors and actuators demands high accuracy, because they are very delicate electromechanical devices. To ensure the highest possible accuracy in the analysis, accurate 3D modeling and analysis have to be employed, especially as stress analysis for durability assessment has become an important issue with the increasing applications of piezoelectric materials.

In the realm of 3D analysis, the boundary integral equation/boundary element method (BIE/BEM), pioneered in the early work [4] for elasticity problems, has been demonstrated to be a viable alternative to the finite element method (FEM) for many problems, due to its features of surface-only discretization and high accuracy in stress and fracture analyses [5–8]. Another advantage of the BIE/BEM, which was recognized only in recent years, is its high accuracy and efficiency in handling thin-body problems, such as thin shell-like structures, layered structures (multi-coatings or thin films), thin voids or open cracks [9–16]. It has been demonstrated that the BIE/BEM can handle the various thin-body problems very effectively, regardless of the thinness of the structures or voids, or non-uniform thickness, as long as the nearly singular integrals are computed accurately [11,12,17]. Much fewer boundary elements can be used to solve these problems for which the number of required finite elements is at least two-orders larger to achieve the same accuracy in stress analysis [11–14]. Considering the fact that the piezoelectric sensors and actuators are often made in thin shapes (films or patches), the BIE/BEM with thin body capabilities has the potential to provide a very efficient and accurate tool in the analysis of such piezoelectric materials.

There have been increasing research efforts in the analysis of piezoelectric materials by the BEM in recent years, as the advantages of the BEM for such analysis is being recognized. For piezoelectric solids without defects, Barnett and Lothe [18] derived a 2D fundamental solution for anisotropic piezoelectric solids. Meric and Saigal [19] derived integral

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formulations for shape sensitivity analysis of 3D piezoelectric solids. The formulations were tested on a 1D problem. Lee and Jiang [20–22] developed the first BIE formulation for the piezoelectric solids. The 2D BEM was implemented and tested for an infinite piezoelectric medium with a cylindrical hole under mechanical and electric loads [22]. Lu and Mahrenholtz [23] derived a variational BEM formulation for piezoelectric solids, which yields symmetric matrices. However, no numerical implementation and examples were given in Ref. [23]. A 3D BEM for piezoelectric solids was first developed by Chen and Lin [24]. The BEM formulation was based on the fundamental solutions derived earlier by Chen [25,26] for 3D piezoelectric solids. The numerical examples using linear elements on a piezoelectric cube and a spherical cavity were presented in Ref. [24]. Wang [27] derived the explicit expressions of the 2D fundamental solutions for piezoelectric materials. Dunn and Wienecke [28] also derived the closed-form expressions for the fundamental solution for transversely isotropic piezoelectric solids. Hill and Farris [29] applied the quadratic (eight-node) boundary elements for 3D piezoelectric bodies and tested their approach on the cube and spherical void problems. Ding et al. [30,31] derived the fundamental solutions in terms of harmonic functions and developed the BEM with several test cases for 2D [30] and 3D problems [31]. Recently, Jiang [32] derived the fundamental solutions and the BIE for 3D time-dependent thermo-piezoelectricity. No numerical examples were given in Ref. [32] for this very complicated case.

For piezoelectric solids with defects (various voids and cracks), Xu and Rajapakse [33] studied the influence and interactions of various holes in 2D piezoelectric media using a coupled BEM. Zhao et al. [34,35] derived the 3D fundamental solutions and the BIE for a penny-shaped crack in a piezoelectric solid. Pan [36] recently presented a detailed study on cracks in 2D piezoelectric media using the BEM. Both the conventional BIE and a hypersingular BIE (traction BIE) were employed in Ref. [36] to handle the possible degeneracy of the BIEs for crack problems. Numerical results in Ref. [36] show excellent agreement between the BEM and the analytical solutions. Recently, Qin [37] studied the interactions of cracks in a piezoelectric half-plane and under thermal loading using the BEM.

All the above results have clearly demonstrated the accuracy and efficiency of the BEM, especially in stress and fracture analyses, for single and bulky piezoelectric materials. However, there are many confusions and unanswered questions regarding the BIE formulation for piezoelectric solids. For examples, how to evaluate the jump terms and thus the free term in the BIE (this is not trivial as in the elasticity case, since the fundamental solutions in the piezoelectricity case are in general not available in explicit forms)? How to compute the singular integrals in the BIE? Or, is there a weakly singular form of the BIE as in the case of elasticity? Are there any integral identities satisfied by the piezoelectric fundamental solutions? Will the piezoelectric BIE degenerate or not when it is applied to crack or thin shell-like problems? The same results as in the case of elasticity BIE, such as the values of the jump terms in the limit as the source point approaches the boundary and the free terms in the BIE, have been assumed in the derivations of the piezoelectric BIE in all the reported work [21,22,24,29,30,36]. Although they turn out to be correct, as will be proved in this paper, the piezoelectric BIE needs special attention since the elasticity BIE is only a special case (subset) of the piezoelectric BIE which can have different properties due to the presence of the electric field. Results in the elasticity BIE (a special case) should not be generalized directly to the piezoelectric BIE (a more general case).

In this paper, we will address the questions and issues raised in the above paragraph. The BIE formulation for piezoelectric solids is revisited first. The key relations employed in the development of the piezoelectric BIE, such as the generalized Green's identity (reciprocal work theorem) and integral identities for the fundamental solution, are derived carefully. A new weakly singular form of the BIE is developed using the identities derived, which can eliminate the calculation of any singular integrals in the discretizations of the BIE using the BEM. Then the crucial question of whether or not the piezoelectric BIE will degenerate when applied to crack and thin shell-like structures is investigated. It is shown that the conventional BIE for piezoelectricity does degenerate for crack problems, but does not degenerate for shell-like structures, in the limit as the two opposing surfaces approaching each other. The latter has significant implications in the applications of the piezoelectric BIE to piezoelectric films used widely as sensors and actuators in smart materials. All the above results for the piezoelectric BIE are consistent with those for the conventional BIE in elasticity. The detailed derivations of the piezoelectric BIE and the proof regarding the degeneracy presented in this paper will clear the confusions in the literature on the piezoelectric BIE and thus establish the BIE/BEM approach on a solid theoretical ground. Numerical tests to show the degeneracy of the piezoelectric BIE for crack problems are presented and one remedy to this degeneracy by using the multi-domain BEM is also demonstrated.

2. The boundary integral equation (BIE) formulation for piezoelectricity

2.1. Governing equations in piezoelectricity

Consider a piezoelectric solid occupying a 3D domain V with the boundary S, Fig. 1. The basic equations governing the elastic and electric fields in a linear piezoelectric material can be summarized in the following (see, e.g. Refs. [1,21,24]) (index notation is used in this paper).

Equilibrium equations:

$$\sigma_{ij,j} + f_i = 0, \tag{1}$$

$$D_{i,i} - q = 0, \tag{2}$$

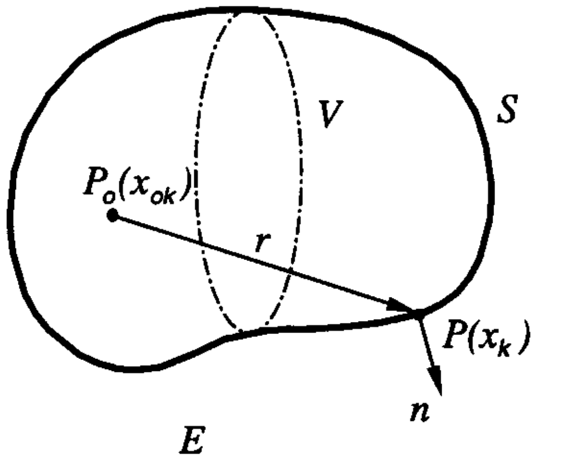

Fig. 1. Domain V in R^3 with boundary S (exterior domain $E = R^3 - (V \cup S)$).

where σ_{ij} is the stress tensor, f_i the body force vector per unit volume, D_i the electric displacement vector and q the intrinsic electric charge per unit volume.

Constitutive equations:

$$\sigma_{ij} = C_{ijkl}s_{kl} - e_{kij}E_k, \qquad \text{(converse effect)} \tag{3}$$

$$D_i = e_{ikl}s_{kl} + \varepsilon_{ik}E_k, \qquad \text{(direct effect)} \tag{4}$$

where s_{kl} is the strain tensor, E_k the electric field, C_{ijkl} the elastic modulus tensor measured in a constant electric field, e_{ijk} the piezoelectric tensor and ε_{ij} the dielectric tensor measured at constant strains.

Strain and electric fields:

$$s_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{5}$$

$$E_i = -\phi_{,i}, \tag{6}$$

where u_i is the elastic displacement vector and ϕ the electric potential.

Boundary conditions (BCs):

$$t_i = \sigma_{ij}n_j = \bar{t}_i, \text{ on } S_t, \quad u_i = \bar{u}_i, \text{ on } S_u; \qquad \text{(mechanical BCs)} \tag{7}$$

$$\omega = -D_i n_i = \bar{\omega}, \ on\ S_\omega, \quad \phi = \bar{\phi}, \ on\ S_\phi; \qquad \text{(electric BCs)} \tag{8}$$

where t_i is the traction, ω the surface charge, n_i the unit outward normal vector (Fig. 1) and the barred quantities indicate given values. Note that the boundary $S = S_t \cup S_u = S_\omega \cup S_\phi$.

Eqs. (1)–(6) under boundary conditions (7) and (8) form the complete mathematical description of the coupled elastic and electric fields in a general anisotropic piezoelectric solid. For an isotropic elastic material, there is no coupling of the elastic and electric fields, that is, the piezoelectric tensor $e_{ijk} = 0$. In this case, Eqs. (1)–(6) will be decoupled between the two fields, yielding the usual elasticity equations and a Poisson's equation for the electric potential ϕ.

2.2. Generalized Green's identity

We first establish the following generalized Green's identity, or reciprocal work theorem, for the piezoelectric solids:

$$\begin{aligned}\int_S t_i u_i^* \,\mathrm{d}S + \int_V f_i u_i^* \,\mathrm{d}V + \int_S \omega^* \phi \,\mathrm{d}S + \int_V q^* \phi \,\mathrm{d}V \\ = \int_S t_i^* u_i \,\mathrm{d}S + \int_V f_i^* u_i \,\mathrm{d}V + \int_S \omega\phi^* \,\mathrm{d}S + \int_V q\phi^* \,\mathrm{d}V,\end{aligned} \tag{9}$$

in which $u_i, t_i, \phi, \omega, \ldots$, and $u_i^*, t_i^*, \phi^*, \omega^*, \ldots$ are two sets of admissible solutions satisfying Eqs. (1)–(8).

To prove identity (9), we first prove the following relation of the internal energy densities for piezoelectric solids:

$$(\sigma_{ij} + e_{kij}E_k)s_{ij}^* = (\sigma_{ij}^* + e_{kij}E_k^*)s_{ij}. \tag{10}$$

Applying Eq. (3), and noting the symmetries in the material constant tensors:

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \qquad e_{kij} = e_{kji}, \qquad \varepsilon_{ik} = \varepsilon_{ki}, \tag{11}$$

we can derive,

$$\begin{aligned}(\sigma_{ij} + e_{kij}E_k)s_{ij}^* &= (C_{ijkl}s_{kl})s_{ij}^* = (C_{klij}s_{kl})s_{ij}^* = (C_{ijkl}s_{ij})s_{kl}^* \\ &= (C_{ijkl}s_{kl}^*)s_{ij} = (\sigma_{ij}^* + e_{kij}E_k^*)s_{ij},\end{aligned}$$

which proves relation (10). Note that the term $(\sigma_{ij} + e_{kij}E_k)$ represents the mechanical stresses, as shown from Eq. (3). Thus, Eq. (10) is a statement of the equivalence of the (virtual) mechanical strain energy densities.

Now, integrating the left hand side of Eq. (10) over the domain V, and applying Eqs. (1)–(8) and the Gauss theorem, we have,

$$\begin{aligned}\int_V (\sigma_{ij} + e_{kij}E_k)s_{ij}^* \,\mathrm{d}V &= \int_V \sigma_{ij}u_{i,j}^* \,\mathrm{d}V + \int_V (e_{ikl}s_{kl}^*)E_i \,\mathrm{d}V \\ &= \int_V (\sigma_{ij}u_i^*)_{,j} \,\mathrm{d}V - \int_V \sigma_{ij,j}u_i^* \,\mathrm{d}V + \int_V (D_i^* - \varepsilon_{ik}E_k^*)E_i \,\mathrm{d}V \\ &= \int_S t_i u_i^* \,\mathrm{d}S + \int_V f_i u_i^* \,\mathrm{d}V - \int_V D_i^* \phi_{,i} \,\mathrm{d}V \\ &\quad - \int_V \varepsilon_{ik}E_i E_k^* \,\mathrm{d}V \\ &= \int_S t_i u_i^* \,\mathrm{d}S + \int_V f_i u_i^* \,\mathrm{d}V - \int_V (D_i^*\phi)_{,i} \,\mathrm{d}V + \int_V D_{i,i}^*\phi \,\mathrm{d}V \\ &\quad - \int_V \varepsilon_{ik}E_i E_k^* \,\mathrm{d}V,\end{aligned}$$

that is,

$$\int_V (\sigma_{ij} + e_{kij}E_k)s^*_{ij}\,\mathrm{d}V$$
$$= \int_S t_i u^*_i\,\mathrm{d}S + \int_V f_i u^*_i\,\mathrm{d}V + \int_S \omega^*\phi\,\mathrm{d}S$$
$$+ \int_V q^*\phi\,\mathrm{d}V - \int_V \varepsilon_{ik}E_iE^*_k\,\mathrm{d}V. \quad (12)$$

Similarly, integrating the right hand side of Eq. (10), we have,

$$\int_V (\sigma^*_{ij} + e_{kij}E^*_k)s_{ij}\,\mathrm{d}V$$
$$= \int_S t^*_i u_i\,\mathrm{d}S + \int_V f^*_i u_i\,\mathrm{d}V + \int_S \omega\phi^*\,\mathrm{d}S$$
$$+ \int_V q\phi^*\,\mathrm{d}V - \int_V \varepsilon_{ik}E^*_iE_k\,\mathrm{d}V. \quad (13)$$

Note that $\varepsilon_{ik}E_iE^*_k = \varepsilon_{ik}E^*_iE_k$ due to the symmetry of ε_{ik} (Eq. (11)).

Thus, integrating Eq. (10) over the domain V and applying results (12) and (13), we obtain the generalized Green's identity (9). Through above derivation, one can clearly identify the physical meanings and the sources of each term in identity (9). Results (12) and (13) are statements of the balance of the (virtual) work done by external forces/charges and the internal strain energy stored in the piezoelectric solid, while identity (9) is a statement of the equivalence of the work done by the two force/charge systems.

2.3. Fundamental solutions

The fundamental solution for the piezoelectric problems is the responses due to independently applied sources. One source is a unit concentrated force in one of the coordinate directions and the other is a unit concentrated charge, at the source point in an infinite piezoelectric medium.

First, consider the responses at a field point P due to a unit concentrated force acting at the source point P_0 and in the direction i ($i = 1,2,3$). The equilibrium equations are,

$$\Sigma_{ijk,k}(P,P_0) + \delta_{ij}\delta(P,P_0) = 0, \qquad \Delta_{ik,k}(P,P_0) = 0, \quad (14)$$

in which Σ_{ijk} and Δ_{ik} are the stress and electric displacement in the fundamental solution, respectively, δ_{ij} the Kronecker δ symbol, $\delta(P,P_0)$ the Dirac δ-function, and the indices $i,j,k,\ldots = 1,2,3$. The displacement, traction, electric potential and the surface charge in this case are denoted, respectively, by

$$U_{ij}(P,P_0),\ \ T_{ij}(P,P_0),\ \ \Phi_i(P,P_0),\ \ \Omega_i(P,P_0),$$

$i,j = 1,2,3.$

Next, consider the responses at P due to a unit charge acting at P_0. The equilibrium equations are,

$$\Sigma_{4jk,k}(P,P_0) = 0, \qquad \Delta_{4k,k}(P,P_0) - \delta(P,P_0) = 0, \quad (15)$$

in which Σ_{4jk} and Δ_{4k} ($j,k = 1,2,3$) are the stress and electric displacement, respectively. The displacement, traction, electric potential and the surface charge in this case are denoted, respectively, by

$$U_{4j}(P,P_0),\ \ T_{4j}(P,P_0),\ \ \Phi_4(P,P_0),\ \ \Omega_4(P,P_0),$$

$j = 1,2,3.$

Here the index "4" is used to indicate that the responses are due to the unit charge.

In general, the fundamental solution for piezoelectric solids can not be expressed in explicit forms yet [24,25,29], except for some special cases, such as transversely isotropic piezoelectric solids or 2D cases [21,22,27,28,30,31,36]. However, the order of singularity of the fundamental solution has been found to be the same as that of the fundamental solution for elasticity problems. For example, in 3D, the displacement is of order $O(1/r)$ (weakly singular), and the stress or traction is of order $O(1/r^2)$ (strongly singular), with r being the distance from the source point to the field point (Fig. 1).

2.4. Identities for the fundamental solution

We now derive some identities satisfied by the piezoelectric fundamental solution, which will be very useful later in establishing the weakly singular BIE, as demonstrated in the context of potential and elasticity problems [38–40].

The following sifting property of the Dirac δ-function [41,42] will be applied frequently (Fig. 1)

$$\int_S F(P)\delta(P,P_0)\,\mathrm{d}S(P) = \begin{cases} F(P_0), & \forall P_0 \in V, \\ \frac{1}{2}F(P_0), & \forall P_0 \in S(\text{smooth}), \\ 0, & \forall P_0 \in E, \end{cases} \quad (16)$$

where $F(P)$ is any continuous function, including $F(P) =$ constants. The integral for $P_0 \in S$ is a Cauchy principal value (CPV) integral and can be verified by considering the physical meaning of the Dirac δ-function (e.g. a unit concentrated force at P_0) [42].

Integrating the first equation in Eq. (14) over the domain V (an arbitrary closed domain; see Fig. 1), and applying the Gauss theorem and formula (16), we obtain,

$$\int_S T_{ij}(P,P_0)\,\mathrm{d}S(P) = \begin{cases} -\delta_{ij}, & \forall P_0 \in V, \\ -\frac{1}{2}\delta_{ij}, & \forall P_0 \in S(\text{smooth}), \\ 0, & \forall P_0 \in E, \end{cases} \quad (17)$$

where $T_{ij} = \Sigma_{ijk}n_k$ has been applied and the indices $i,j = 1,2,3$. Integrating the second equation in Eq. (14), we obtain another identity,

$$\int_S \Omega_i(P,P_0)\,\mathrm{d}S(P) = 0, \qquad \forall P_0 \in V \cup S \cup E, \tag{18}$$

where $\Omega_i = -\Delta_{ik} n_k$ has been applied and $i = 1,2,3$.

Similarly, integrating the two equations in Eq. (15), we obtain another two identities,

$$\int_S T_{4j}(P,P_0)\,\mathrm{d}S(P) = 0, \qquad \forall P_0 \in V \cup S \cup E, \tag{19}$$

$$\int_S \Omega_4(P,P_0)\,\mathrm{d}S(P) = \begin{cases} -1, & \forall P_0 \in V, \\ -\frac{1}{2}, & \forall P_0 \in S(\text{smooth}), \\ 0, & \forall P_0 \in E, \end{cases} \tag{20}$$

where $j = 1,2,3$.

The established integral identities (17)–(20) for the piezoelectric fundamental solution represent the equilibrium of the forces or charges over boundary S of body V in the presence of the unit force or charge (cf. results in Refs. [38–40] for potential and elasticity problems).

2.5. Weakly singular BIE formulation for piezoelectricity

To derive the BIE, we first choose, in the generalized Green's identity (9) (with the dummy index i being replaced by j), the field

$$u_j^* = U_{ij}, \;\; t_j^* = T_{ij}, \;\; \phi^* = \Phi_i, \;\; \omega^* = \Omega_i,$$

$$f_j^* = \delta_{ij}\delta(P,P_0), \;\; \text{and } q^* = 0,$$

to be the fundamental solution due to the unit force, while the field u_j, t_j, ϕ, ω and q to be the solution satisfying Eqs. (1)–(8). We have from identity (9),

$$\int_S t_j U_{ij}\,\mathrm{d}S + \int_V f_j U_{ij}\,\mathrm{d}V + \int_S \Omega_i \phi\,\mathrm{d}S$$
$$= \int_S T_{ij} u_j\,\mathrm{d}S + \int_V \delta_{ij}\delta(P,P_0) u_j\,\mathrm{d}V + \int_S \omega \Phi_i\,\mathrm{d}S$$
$$+ \int_V q\Phi_i\,\mathrm{d}V.$$

Using Eq. (16) and identifying the variables explicitly, we obtain the following *representation integral for the displacement field*,

$$\begin{aligned} u_i(P_0) &= \int_S U_{ij}(P,P_0)t_j(P)\,\mathrm{d}S(P) - \int_S T_{ij}(P,P_0)u_j(P)\,\mathrm{d}S(P) \\ &\quad - \int_S \Phi_i(P,P_0)\omega(P)\,\mathrm{d}S(P) + \int_S \Omega_i(P,P_0)\phi(P)\,\mathrm{d}S(P) \\ &\quad + \int_V U_{ij}(P,P_0)f_j(P)\,\mathrm{d}V(P) - \int_V \Phi_i(P,P_0)q(P)\,\mathrm{d}S(P), \\ &\forall P_0 \in V, \end{aligned} \tag{21}$$

in which $i,j = 1,2,3$.

Similarly, if we choose, in identity (9),

$$u_j^* = U_{4j}, \;\; t_j^* = T_{4j}, \;\; \phi^* = \Phi_4, \;\; \omega^* = \Omega_4, \;\; f_j^* = 0,$$

and $q^* = \delta(P,P_0)$,

to be the fundamental solution due to the unit charge, we obtain the following *representation integral for the electric potential field*,

$$\begin{aligned} -\phi(P_0) &= \int_S U_{4j}(P,P_0)t_j(P)\,\mathrm{d}S(P) - \int_S T_{4j}(P,P_0)u_j(P)\,\mathrm{d}S(P) \\ &\quad - \int_S \Phi_4(P,P_0)\omega(P)\,\mathrm{d}S(P) + \int_S \Omega_4(P,P_0)\phi(P)\,\mathrm{d}S(P) \\ &\quad + \int_V U_{4j}(P,P_0)f_j(P)\,\mathrm{d}V(P) - \int_V \Phi_4(P,P_0)q(P)\,\mathrm{d}S(P), \\ &\forall P_0 \in V, \end{aligned} \tag{22}$$

in which $j = 1,2,3$.

From the two representation integrals (21) and (22), we can clearly identify the similarities as compared with the elasticity case and the coupling between the displacement and electric fields. To simplify the notation to make it easier for the numerical implementation, we adopt the following matrix notation [30],

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ -\phi \end{Bmatrix}, \qquad \mathbf{t} = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ -\omega \end{Bmatrix}, \qquad \mathbf{b} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ -q \end{Bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} & \Phi_1 \\ U_{21} & U_{22} & U_{23} & \Phi_2 \\ U_{31} & U_{32} & U_{33} & \Phi_3 \\ U_{41} & U_{42} & U_{43} & \Phi_4 \end{bmatrix}, \tag{23}$$

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & \Omega_1 \\ T_{21} & T_{22} & T_{23} & \Omega_2 \\ T_{31} & T_{32} & T_{33} & \Omega_3 \\ T_{41} & T_{42} & T_{43} & \Omega_4 \end{bmatrix}.$$

Then, the representation integrals (21) and (22) can be combined to yield,

$$\begin{aligned} \mathbf{u}(P_0) &= \int_S \mathbf{U}(P,P_0)\mathbf{t}(P)\,\mathrm{d}S(P) - \int_S \mathbf{T}(P,P_0)\mathbf{u}(P)\,\mathrm{d}S(P) \\ &\quad + \int_V \mathbf{U}(P,P_0)\mathbf{b}(P)\,\mathrm{d}V(P), \\ &\forall P_0 \in V, \end{aligned} \tag{24}$$

where **u**, **t** and **b** can be called the generalized (or extended) displacement, traction and body force vectors, respectively;

and **U** and **T** the generalized displacement and traction kernels, respectively.

Before we let the source point P_0 go to the boundary in Eq. (24) to derive the BIE, we note that the integral with the strongly singular kernel **T** can be regularized by using the identities (17)–(20) which can now be written in the following matrix form:

$$\int_S \mathbf{T}(P,P_0)\,\mathrm{d}S(P) = \begin{cases} -\mathbf{I}, & \forall P_0 \in V, \\ -\frac{1}{2}\mathbf{I}, & \forall P_0 \in S(\text{smooth}), \\ \mathbf{0}, & \forall P_0 \in E, \end{cases} \tag{25}$$

where **I** is a 4×4 identity matrix. Note that the first part of this identity (for $P_0 \in V$) can also be derived by considering a simple solution $\mathbf{u}(P) = \mathbf{I}$, which satisfies the governing Eqs. (1)–(6) with $f_i = 0$ and $q = 0$, in the representation integral (24) (cf. the results in Refs. [38–40]).

Therefore, for the integral with the strongly singular kernel **T** in Eq. (24), we have

$$\begin{aligned} \int_S \mathbf{T}(P,P_0)\mathbf{u}(P)\,\mathrm{d}S(P) &= \int_S \mathbf{T}(P,P_0)[\mathbf{u}(P)-\mathbf{u}(P_0)]\,\mathrm{d}S(P) \\ &\quad + \int_S \mathbf{T}(P,P_0)\,\mathrm{d}S(P)\mathbf{u}(P_0) \\ &= \int_S \mathbf{T}(P,P_0)[\mathbf{u}(P)-\mathbf{u}(P_0)]\,\mathrm{d}S(P) - \mathbf{u}(P_0), \\ &\forall P_0 \in V, \end{aligned} \tag{26}$$

by using identity (25) (cf. the potential and elasticity cases [38–40]).

Substituting result (26) into Eq. (24), and letting the source point P_0 go to the boundary S, we obtain the following *weakly singular form of the BIE* in piezoelectricity,

$$\begin{aligned} &\int_S \mathbf{T}(P,P_0)[\mathbf{u}(P)-\mathbf{u}(P_0)]\,\mathrm{d}S(P) \\ &= \int_S \mathbf{U}(P,P_0)\mathbf{t}(P)\,\mathrm{d}S(P) + \int_V \mathbf{U}(P,P_0)\mathbf{b}(P)\,\mathrm{d}V(P), \\ &\forall P_0 \in S, \end{aligned} \tag{27}$$

for a *finite* domain (interior problem). There are no jump terms arising from the limit process as the source point P_o goes to the boundary S, since all integrals involved are at most weakly singular, e.g. of order $O(1/r)$ for 3D problems, after the regularization shown in Eq. (26).

Similarly, for an *infinite* domain (exterior problem), we can establish the following *weakly singular form of the BIE*

$$\begin{aligned} &\mathbf{u}(P_0) + \int_S \mathbf{T}(P,P_0)[\mathbf{u}(P)-\mathbf{u}(P_0)]\,\mathrm{d}S(P) \\ &= \int_S \mathbf{U}(P,P_0)\mathbf{t}(P)\,\mathrm{d}S(P) + \int_V \mathbf{U}(P,P_0)\mathbf{b}(P)\,\mathrm{d}V(P), \\ &\forall P_0 \in S. \end{aligned} \tag{28}$$

The weakly singular BIE (27) or (28) for piezoelectric solids has several advantages, compared with the following singular BIE in the literature:

$$\begin{aligned} &\mathbf{C}(P_0)\mathbf{u}(P_0) + \int_S \mathbf{T}(P,P_0)\mathbf{u}(P)\,\mathrm{d}S(P) \\ &= \int_S \mathbf{U}(P,P_0)\mathbf{t}(P)\,\mathrm{d}S(P) + \int_V \mathbf{U}(P,P_0)\mathbf{b}(P)\,\mathrm{d}V(P), \\ &\forall P_0 \in S, \end{aligned} \tag{29}$$

where **C** is a coefficient matrix depending on the smoothness of S at P_0 (see next section).

First, there are no singular integrals in the weakly singular BIE and its discretization leads directly to the conclusion that the diagonal terms can be determined by summing the off-diagonal terms for the matrix involving the singular kernel **T** [39]. Second, by employing the identity for the piezoelectric fundamental solution, we do not have to evaluate any jump terms explicitly in deriving the weakly singular BIE (27) or (28). Evaluations of the jump terms are required in deriving the singular BIE (29) as used in the literature. For general 3D piezoelectric solids, the fundamental solution is not available in explicit form and the jump terms have been assume to be the same as in the elasticity BIE, without sufficient justifications, in the literature. The use of the weakly singular BIE can avoid the confusions caused by this lack of justifications. Third, regularizing the singular integrals in the BIEs has become a standard approach to dealing with the singular integrals in the BEM [43,44]. The weakly singular nature of the piezoelectric BIE is quite general, as in the cases for potential and elastostatic problems [38–40]. Not only can the various strongly singular (conventional) BIEs be recast in weakly singular forms, but also can the hypersingular BIEs be written in weakly singular forms, by employing the various identities for the fundamental solutions [39,40] or through other means.

3. Degeneracy issues with the piezoelectric BIE for thin shapes

In this section, we prove that the piezoelectric BIE does degenerate when applied to the two opposing surfaces of a *crack*, but does not degenerate when applied to the two surfaces of a *thin shell-like structure*. Although the same conclusions have been proved analytically and numerically in the context of the elasticity BIE for cracks (see, e.g. [6,9]) and thin shell-like structures [11] (see also Ref. [45]), they can not be generalized automatically to the more general piezoelectric BIE. For example, the coupling of the elastic and electric fields in the piezoelectricity theory could have an effect in these conclusions, unless we can prove rigorously that the same conclusions hold for the piezoelectric BIE.

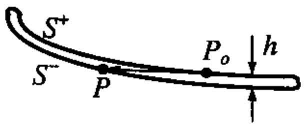

Fig. 2. Boundary S of a thin-shape: $S = S^+ \cup S^-$.

3.1. The jump terms

We need some results for the jump terms in the limit as the source point P_0 approaches the boundary S. We can establish the results for the jump terms readily by applying the identity (25) for the fundamental solution, without exploiting the explicit expressions of the fundamental solution which are not yet available for 3D piezoelectric solids. The results for the jump terms are as follows

$$\lim_{P_0 \to S} \int_S \mathbf{T}(P, P_0)\mathbf{u}(P)\,\mathrm{d}S(P) = \int_S \mathbf{T}(P, P_0)\mathbf{u}(P)\,\mathrm{d}S(P) - \tfrac{1}{2}\mathbf{u}(P_0), \qquad (30)$$

$\forall P_0 \in S$(smooth),

when P_0 approaches S in the *same* direction of the normal n; and

$$\lim_{P_0 \to S} \int_S \mathbf{T}(P, P_0)\mathbf{u}(P)\,\mathrm{d}S(P) = \int_S \mathbf{T}(P, P_0)\mathbf{u}(P)\,\mathrm{d}S(P) + \tfrac{1}{2}\mathbf{u}(P_0), \qquad (31)$$

$\forall P_0 \in S$(smooth),

when P_0 approaches S in the *opposite* direction of the normal n, where the first integral on the right hand side of Eqs. (30) and (31) is a CPV integral.

To prove results (30) and (31), we employ the classical approach in the BEM literature by considering the limit as P_o approaching S from either inside or outside. However, the explicit expressions for kernel $\mathbf{T}$ is not used at all in evaluating these limits. The integral identity (25), which is a concise notation for identities (17)–(20), can be employed to avoid the tedious task of evaluating the integral of the kernel explicitly, which is possible only when explicit expressions for the kernels are available. The details of this new approach in deriving the jump terms (30) and (31), in the context of elastostatics, can be found in Ref. [42].

To study the degeneracy issue with the piezoelectric BIE, we follow the integral operator notation used in Ref. [11]

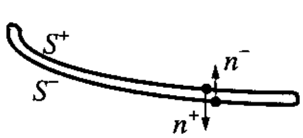

Fig. 3. Boundary and normal for a *crack* (an *exterior* problem).

and write the weakly singular BIE in Eqs. (27) and (28) as

$$D[\mathbf{u}(P) - \mathbf{u}(P_0)] = B\mathbf{t}, \qquad \text{(for an interior problem)} \qquad (32)$$

and

$$\mathbf{u}(P_0) + D[\mathbf{u}(P) - \mathbf{u}(P_0)] = B\mathbf{t}, \qquad (33)$$

(for an exterior problem)

where the body forces and charges have been ignored. The two operators are defined by

$$B[\cdot] = \int_S \mathbf{U}(P, P_0)[\cdot]\,\mathrm{d}S(P), \quad \text{and}$$
$$D[\cdot] = \int_S \mathbf{T}(P, P_0)[\cdot]\,\mathrm{d}S(P), \qquad (34)$$

with $\mathbf{U}$ and $\mathbf{T}$ being the two 4×4 matrices given in Eq. (23).

Consider a thin shape as shown in Fig. 2 and apply the piezoelectric BIE (32) or (33) on both S^+ and S^-. In the limit as the thickness h goes to zero (i.e. S^- goes to S^+), we arrive at different conclusions regarding the degeneracy of the BIE for cracks and for thin shells.

3.2. Degeneracy of the piezoelectric BIE for crack problems

Following the steps in Ref. [11] for the elasticity BIE, applying BIE (33) on the two surfaces of a crack (an *exterior*-like problem, Fig. 3) and employing the jump term relation (31), we can derive the following two equations in the limit as the distance $h \to 0$,

$$D^+(\mathbf{u}^+ - \mathbf{u}^-) + \tfrac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-) = B^+(\mathbf{t}^+ + \mathbf{t}^-), \qquad (35)$$

from $P_0 \in S^+$;

$$D^+(\mathbf{u}^+ - \mathbf{u}^-) + \tfrac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-) = B^+(\mathbf{t}^+ + \mathbf{t}^-), \qquad (36)$$

from $P_0 \in S^-$;

where $\mathbf{u}^+$, $\mathbf{t}^+$, $\mathbf{u}^-$ and $\mathbf{t}^-$ are the generalized displacement and traction vectors, as given in Eq. (23), on S^+ and S^-, respectively, D^+ and B^+ are given by Eq. (34) with S being replaced by S^+ (see Ref. [11]). Eqs. (35) and (36) are exactly the same. Therefore, the piezoelectric BIE *does degenerate* when applied to cracks. The same conclusion can be drawn if we apply the singular BIE (Eq. (29)) instead of the weakly singular BIE (33).

3.3. Non-degeneracy of the piezoelectric BIE for thin piezoelectric shells

Applying BIE (32) on the two surfaces of a thin piezoelectric shell (an *interior*-like problem, Fig. 4) and employing the jump term relation (30), we can obtain the following two equations in the limit as the thickness $h \to 0$,

$$D^+(\mathbf{u}^+ - \mathbf{u}^-) + \tfrac{1}{2}(\mathbf{u}^+ - \mathbf{u}^-) = B^+(\mathbf{t}^+ + \mathbf{t}^-), \qquad (37)$$

from $P_0 \in S^+$;

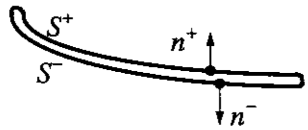

Fig. 4. Boundary and normal for a *thin piezoelectric shell* (an *interior* problem).

$$D^+(\mathbf{u}^+ - \mathbf{u}^-) - \tfrac{1}{2}(\mathbf{u}^+ - \mathbf{u}^-) = B^+(\mathbf{t}^+ + \mathbf{t}^-), \quad (38)$$

from $P_0 \in S^-$;

which are two distinctive equations no matter how thin the shell is, as long as the piezoelectric shell is under realistic boundary conditions (e.g. not constrained on the entire boundary S; see discussions in Ref. [11]). Therefore, the piezoelectric BIE *does not degenerate* when applied to thin shells. Again, the same conclusion can be drawn if we apply the singular BIE (Eq. (29)) instead of the weakly singular BIE (32), as demonstrated for the elasticity BIE in Ref. [11].

4. Numerical examples

To demonstrate that the piezoelectric BIE does degenerate for crack problems, a test problem is considered in this section. A multi-domain BEM approach to remedy this degeneracy problem in using the conventional piezoelectric BIE for crack problems is also presented.

A 2D BEM based on the piezoelectric BIE (27) without body forces and charges, or Eq. (32), is implemented using quadratic (three-node) line elements. Fig. 5 shows a square piezoelectric medium (PZT-4, under plane strain condition) with an elliptical hole at the center. The square domain is sufficiently large compared with the hole ($b/L = 20$) so that the analytical solution in Ref. [46] for an infinite domain can be used to validate the BEM solutions. The materials constants of PZT-4 are as given in Ref. [22] in which only

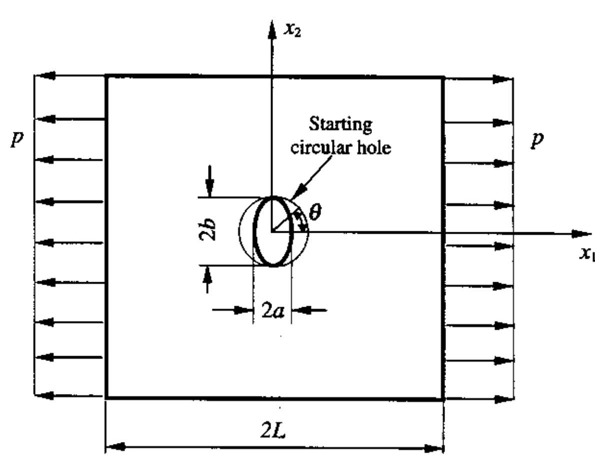

Fig. 5. An elliptical hole in a square piezoelectric medium under tension (plane strain condition).

the circular hole case is considered. The model is under uniform tension in the x_1 direction. The elliptical holes are formed by starting with a circular hole and scaling it in the x_1 direction (axis a).

Figs. 6–8 show the results of the first principal stress, total mechanical displacement and electric displacement at the nodes on the holes, respectively, for three values of the ratio a/b, by the piezoelectric BEM and the analytical solution [46] (Note that the angle θ is measured on the original circular hole in all the cases, Fig. 5). It is shown that the BEM results using only 44 elements are in excellent agreement with the exact solution for all the three quantities in the three cases studied ($a/b = 1.0$, 0.5 and 0.05).

When the ratio a/b is further reduced and the elliptical hole becomes an open crack, many more elements on the edge of the hole are needed in order to obtain possible converged BEM results. Fig. 9 shows the BEM results for the mechanical displacement at the hole when $a/b = 0.01$ with increasing numbers of elements. It is observed that only when the elliptical hole is discretized using 180 elements (a total of 204 elements), do the BEM results converge to the analytical solution. However, when the case $a/b = 0.001$ is studied, even the finest BEM mesh (204 elements) cannot provide converged results, as shown in Fig. 10 for the electric displacement result. Moreover, the symmetry in the BEM results with respect to the crack tip ($\theta = 90°$) is also lost (Fig. 10). This is a clear indication of the degeneracy of the piezoelectric BIE/BEM for crack problems, as predicted by the theory in Section 3.2.

As discussed in Refs. [11,12] and by many others, there are two difficulties when the elasticity BIE/BEM is applied to a thin void (or open crack) with increasingly smaller opening. The first difficulty is in dealing with the nearly singular integrals when the source point is on one surface and the integration on the opposite surface of the crack. The other difficulty is the degeneracy of the BIE when applied to true (zero-opening) cracks. These two difficulties also exist in the piezoelectric BIE/BEM when applied to a thin void in a piezoelectric material, as has been proved and demonstrated in this paper. Increasing the number of elements on the two faces of the open crack (thus decreasing the element sizes) can alleviate the difficulty in computing nearly singular integrals, although it is not the efficient way to deal with nearly singular integrals in the BEM [11,12]. This is why good BEM results are obtained with a large number of elements in the case when $a/b = 0.01$ (Fig. 9). However, increasing the number of elements will not help in easing the difficulty of the BIE/BEM degeneracy for crack problems. This is why the BEM results deteriorate even with the large number of elements in the $a/b = 0.001$ case, which is closer to a true crack (Fig. 10). Alternative BIE formulations or different BEM modeling techniques are needed in order to tackle the crack problems.

Next, the multi-domain BEM approach to crack problems is demonstrated in the context of piezoelectric BIE. The

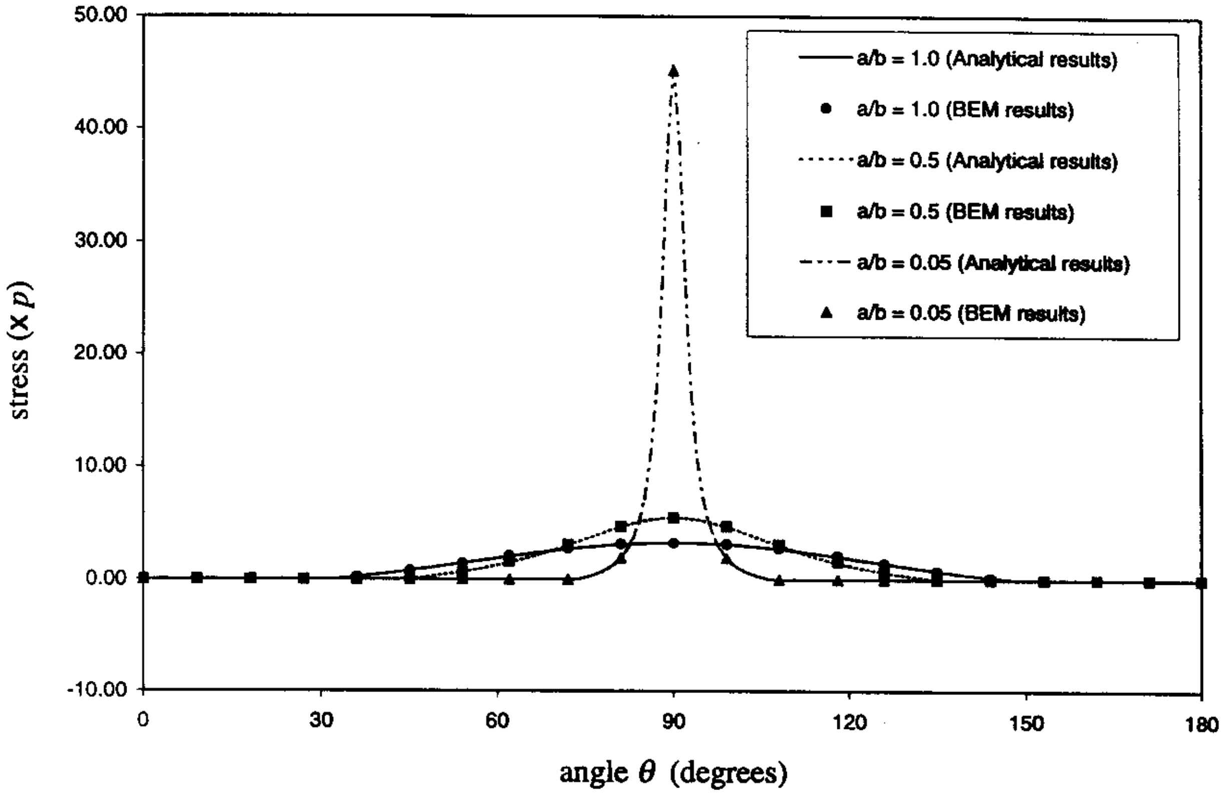

Fig. 6. The first principal stress (σ_1^p) on the edge of the holes (number of boundary elements $M = 44$).

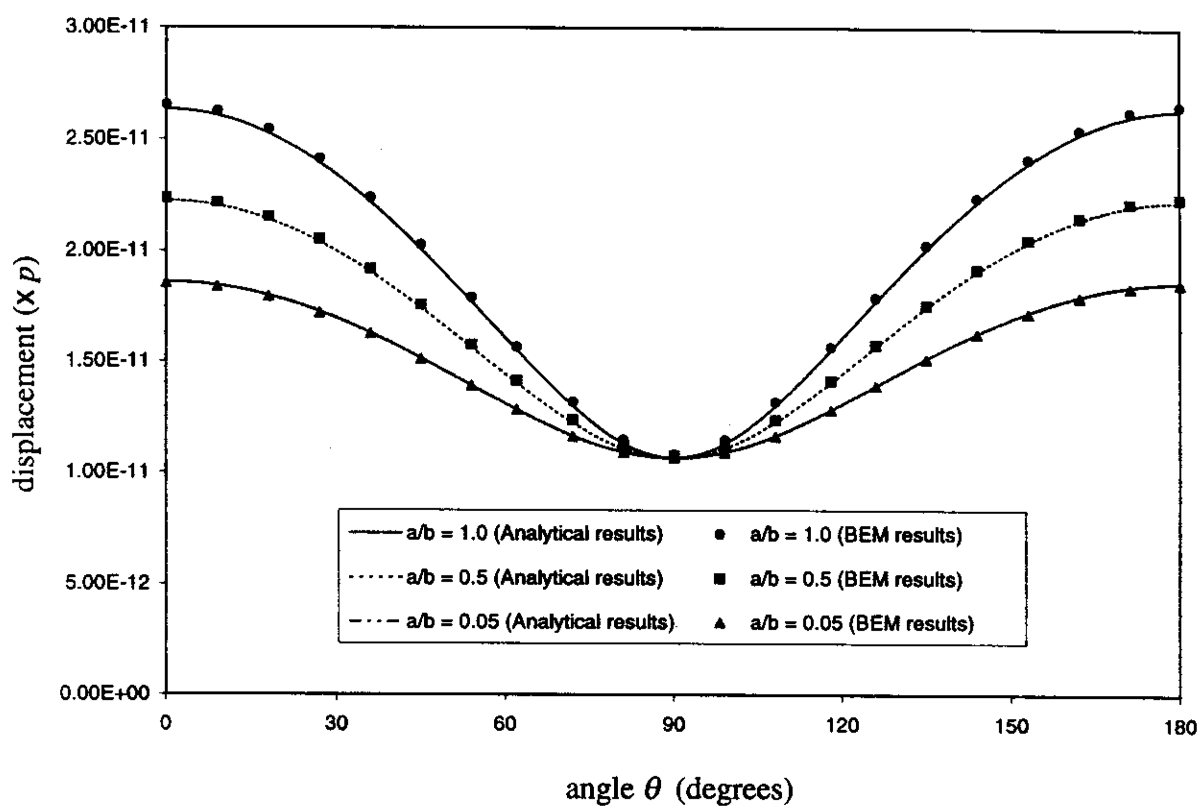

Fig. 7. The total mechanical displacement ($|\mathbf{u}|$) on the edge of the holes (number of boundary elements $M = 44$).

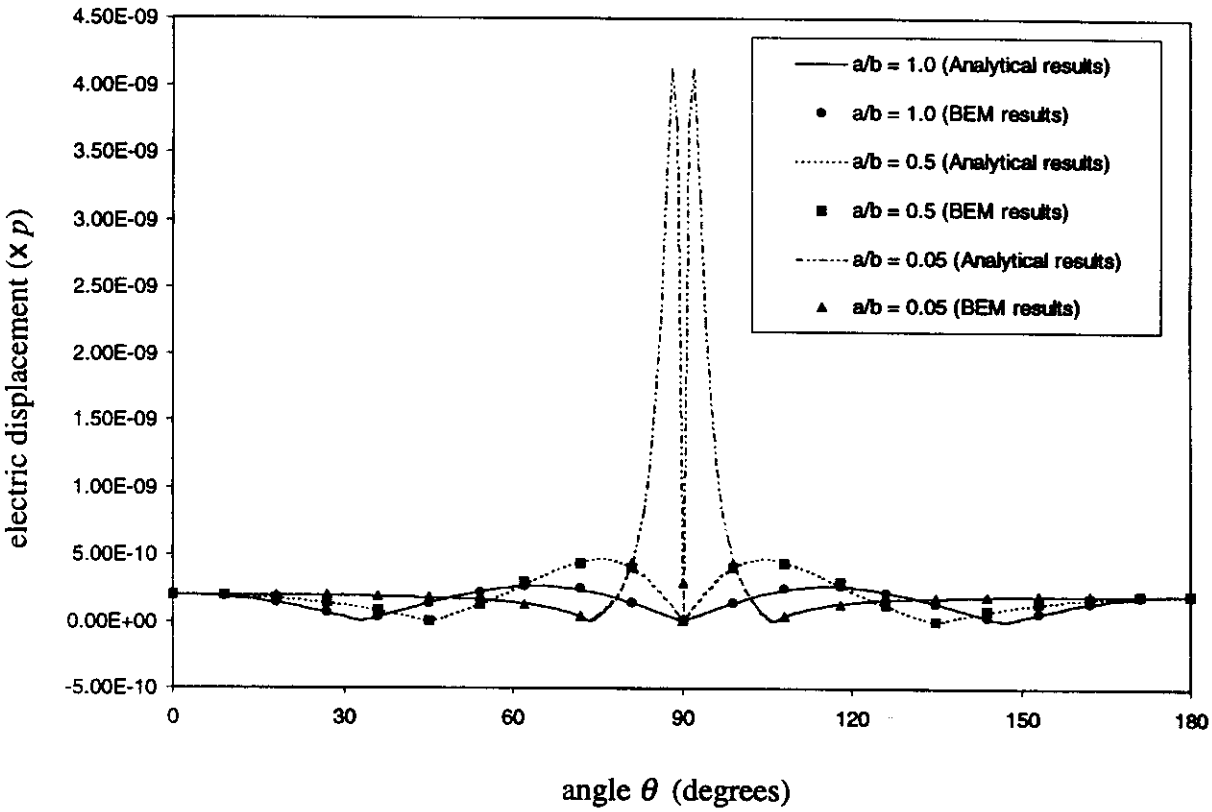

Fig. 8. The magnitude of electric displacement ($|\mathbf{D}|$) on the edge of the holes (number of boundary elements $M = 44$).

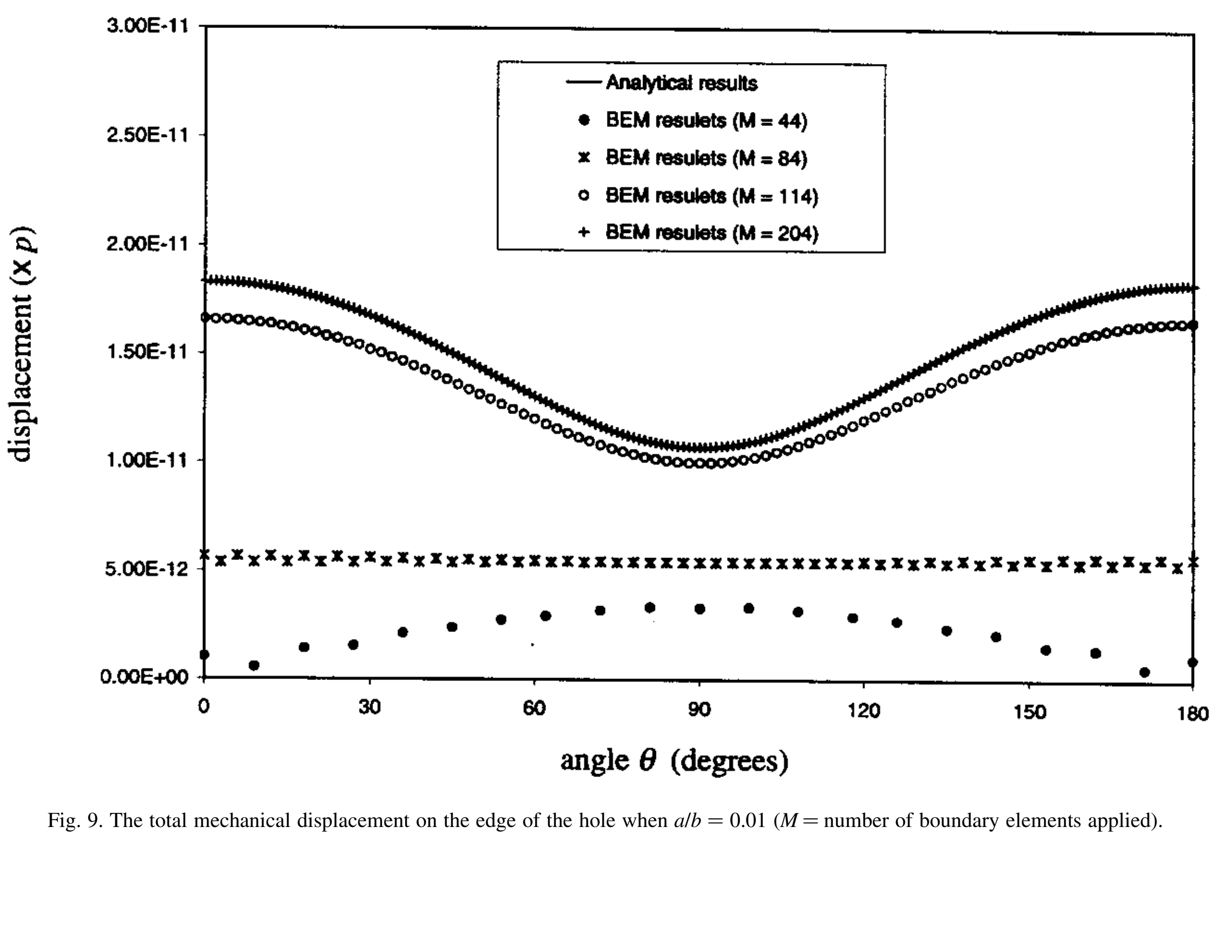

Fig. 9. The total mechanical displacement on the edge of the hole when $a/b = 0.01$ (M = number of boundary elements applied).

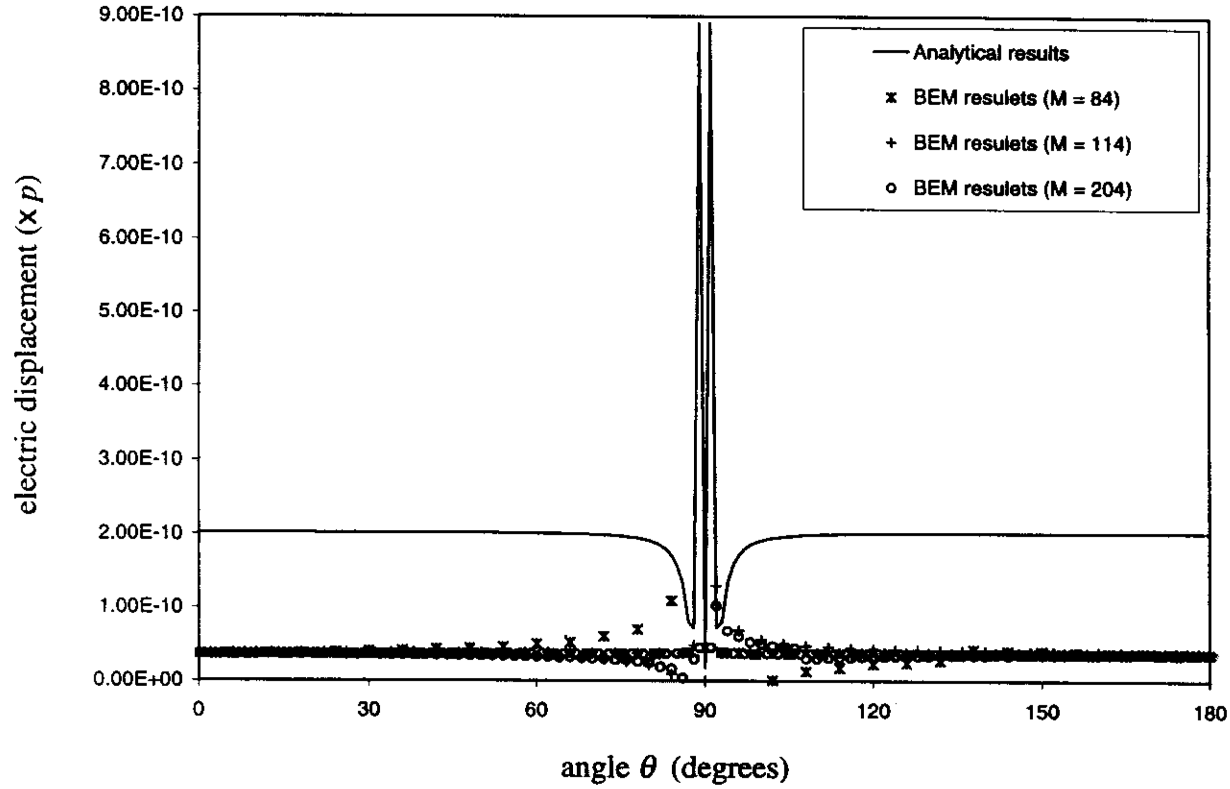

Fig. 10. Degeneracy of the piezoelectric BIE/BEM: results for the magnitude of electric displacement on the edge of the hole when $a/b = 0.001$ ($M =$ number of boundary elements applied).

main idea in the multi-domain BEM for crack problems (see, e.g. Ref. [6]) is to introduce auxiliary interfaces in the domain, starting from the crack tips to the outer boundary of the domain so that the original single domain is divided into two (or several, if needed). Then the conventional BIE is applied to each domain and the two systems of equations are coupled together through the use of interface conditions (e.g. continuity of displacements and equilibrium of stresses). In this way, the degeneracy difficulty, and the nearly singular integrals in some cases, in the BEM for crack problems are avoided since the two equations on the two opposing crack faces are now from different domains. The disadvantage of using this multi-domain approach to crack problems is that the auxiliary interfaces introduced could be large and hence the problem size could be much larger. Nevertheless, it is a simple, straightforward approach to crack problems by using only the conventional BIE.

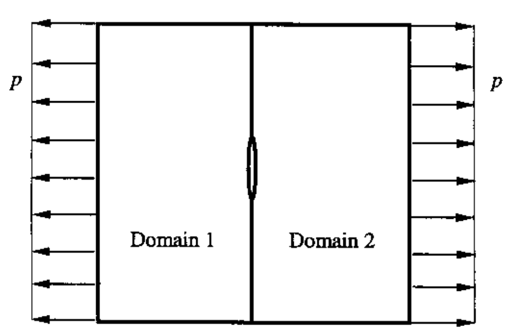

Fig. 11. A multi-domain BEM approach to the crack-like problems.

Fig. 11 shows the division of the piezoelectric medium, considered earlier (see Fig. 5), into two subdomains. Figs. 12–14 show the stress, displacement and electric displacement, respectively, on the edge of the hole (in fact, an open crack) when $a/b = 0.001$ (Fig. 5), using the multi-domain BEM as compared with the single-domain BEM with the same mesh on the hole and the outer boundary. Additional 20 elements are employed on the two interface lines (Fig. 11) for the multi-domain BEM. As shown and discussed earlier (Fig. 10), the single-domain BEM results depart dramatically from the analytical solutions for all the three quantities, due to the degeneracy of the BIE in this open-crack case. However, the multi-domain BEM results agree very well with the analytical solutions with a relatively small number of elements, as expected. Note that the BEM is able to capture the strong singularity of the stresses near the crack tip as shown in Fig. 12. Also note that the hoop stress on the edge of the hole is compressive in most regions in this crack-like case, therefore the first principal stress is zero, except for the regions near the crack tips. The multi-domain BEM results for the electric displacement (Fig. 14) is less accurate in the small region of the crack tip. This is due to the rapid oscillation of the field when $a/b = 0.001$ as shown by the analytical solution.

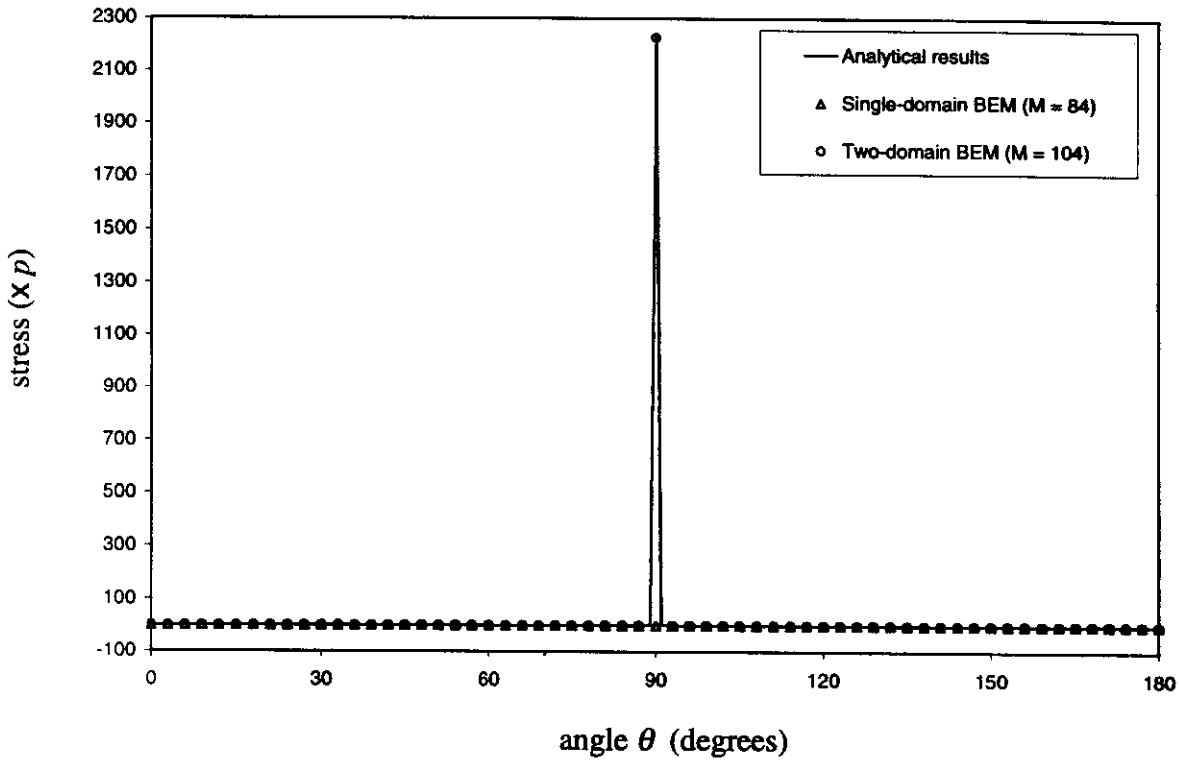

Fig. 12. The first principal stress on the edge of the hole when $a/b = 0.001$ using the multi-domain BEM.

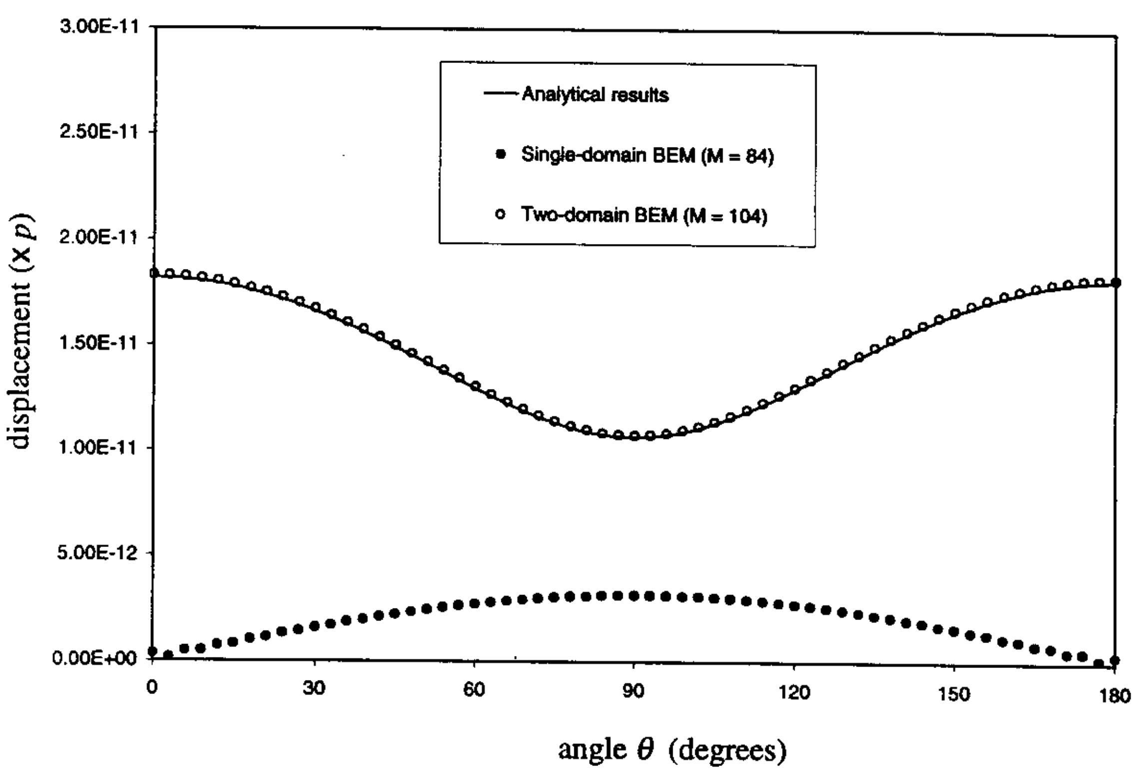

Fig. 13. The total mechanical displacement on the edge of the hole when $a/b = 0.001$ using the multi-domain BEM.

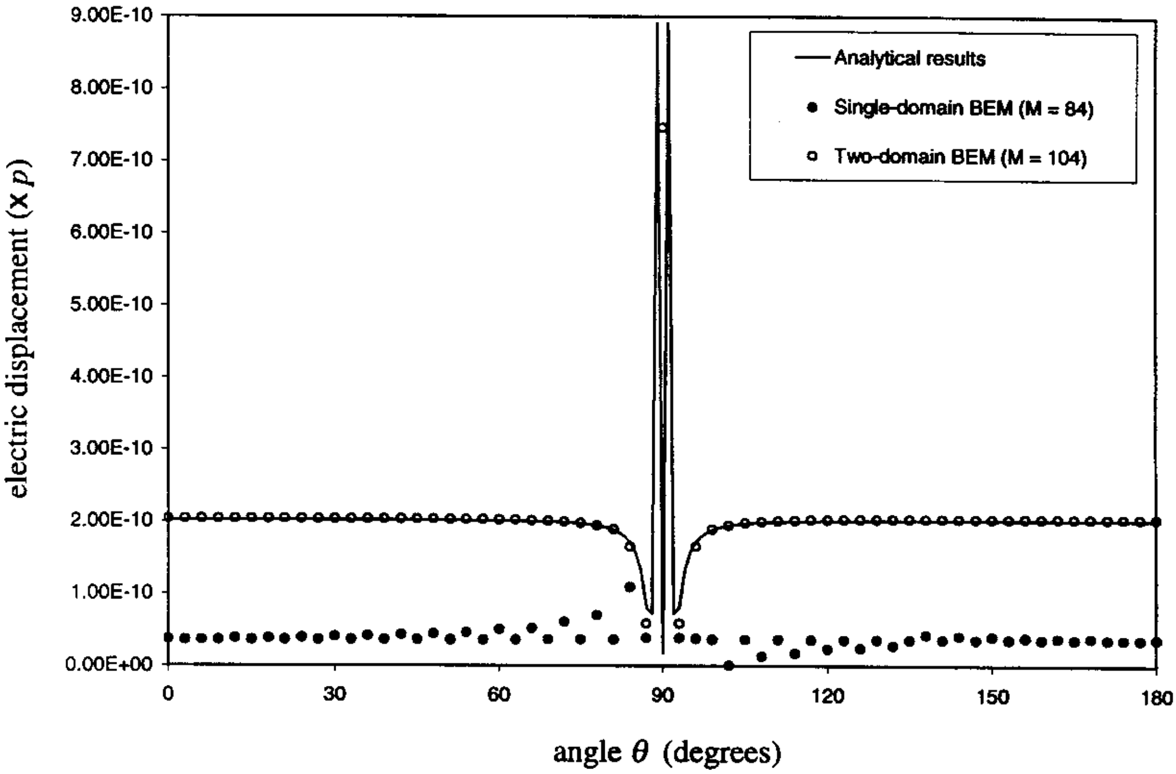

Fig. 14. The magnitude of electric displacement on the edge of the hole when $a/b = 0.001$ using the multi-domain BEM.

The multi-domain piezoelectric BEM based on the conventional BIE, as demonstrated in the above, can provide reasonably good results for the analysis of crack-like problems. However, the more efficient and accurate way to handle crack problems is to apply the piezoelectric hypersingular BIE [36].

5. Discussions

A weakly singular BIE for the analysis of piezoelectric solids has been derived in this paper. The derivation is based on the generalized Green's identity or reciprocal work theorem and the governing equations satisfied by the piezoelectric fundamental solution. The mathematical properties of the Dirac δ-function is exploited fully, which facilitates conciseness and rigorousness in the derivations. Four integral identities for the elastic and electric components in the piezoelectric fundamental solution are established, which are employed in deriving the weakly singular BIE for piezoelectricity. Degeneracy issues with the piezoelectric BIE when applied to thin shapes are examined. It is shown analytically that the piezoelectric BIE does degenerate when applied to cracks, but does not degenerate when applied to thin shells. In deriving or proving all the above results, the explicit expressions of the fundamental solution, which are not yet available for 3D piezoelectric solids, are not needed at all, due to the use of the developed integral identities for the fundamental solution. The established procedures and results for the piezoelectric BIE are general and valid for both 2D and 3D cases. 2D numerical tests to show the degeneracy of the piezoelectric BIE for crack problems are presented and one remedy to this degeneracy by using the multi-domain BEM is also demonstrated in this paper.

Piezoelectric equations are in a more general framework than the elasticity equations. Therefore, results regarding the elasticity BIE, such as values of the jump terms and whether or not degenerate/non-degenerate when applied to thin shapes, can not be generalized to the piezoelectric BIE without explicit proofs. These results have been assumed in the BEM literature for the piezoelectric BIE, which turn out to be correct, but have left many confusions regarding the piezoelectric BIE. This paper serves in part to clarify these confusions by providing the necessary and rigorous derivations or proofs. In the meantime, the method developed in this paper for deriving the BIEs without ever exploiting the explicit expressions of the fundamental solutions is quite interesting and general, which can be applied to other cases when the fundamental solutions are not available in explicit forms.

The *degeneracy* of the piezoelectric BIE when applied to the two surfaces of a crack, as proved in this paper, necessitates the study of the hypersingular BIE for piezoelectric solids with defects. For elastic solids with cracks, the hypersingular (traction) BIE has been found very effective and efficient. However, the results for the hypersingular BIE in elasticity, such as the regularization of the hypersingular kernel, cannot be generalized automatically to the hypersingular BIE in piezoelectricity. Rigorous derivations and proofs are necessary and underway. Results regarding the hypersingular BIE in piezoelectricity, such as additional integral identities for the fundamental solutions, regularization procedures and the weakly singular form, will be reported in a separate paper.

The *non-degeneracy* of the piezoelectric BIE when applied to the two surfaces of a thin piezoelectric shell, also proved in this paper, has significant implications in applications of the piezoelectric BIE to smart materials. The piezoelectric sensors and actuators employed widely in smart material applications are often made in thin films. The BIEs based on elasticity and with thin-body capabilities have been found extremely accurate and efficient in analyzing elastic thin shells, films or coatings [11–15]. Extending these capabilities of the elastic BIE to the piezoelectric BIE promises to provide a much needed accurate and efficient numerical tool for the analysis of piezoelectric sensors and actuators.

The BEM implementation and applications of the developed weakly singular piezoelectric BIE to thin piezoelectric films, including treatment of nearly singular integrals in such applications, are underway and will be reported in a subsequent paper.

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